Mimetic Least Squares Spectral/hp Finite Element Method for the Poisson Equation

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The Standard Least Squares
- How does it work?
- The finite dimensional spaces
- Example: 2D Poisson equation
- Why it does not work?

Mimetic Approach
- Going back to the basics
- Differential geometry
- Mimetic least-squares

Summary and Future Work
- Summary
- Future work
- Further reading

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The principle

The partial differential equation

$$\begin{cases} \mathcal{L}\mathbf{u} &= f \quad \text{in} \quad \Omega \\ \mathcal{R}\mathbf{u} &= h \quad \text{on} \quad \Gamma \end{cases}$$

The principle

The partial differential equation

$$\begin{cases} \mathcal{L}\mathbf{u} &= f \quad \text{in} \quad \Omega \\ \mathcal{R}\mathbf{u} &= h \quad \text{on} \quad \Gamma \end{cases}$$

Reduce the dimension of the problem (discretize)

$$\begin{cases} \mathcal{L}\mathbf{u}_{h,p} &= f \quad \text{in} \quad \Omega \\ \mathcal{R}\mathbf{u}_{n,p} &= h \quad \text{on} \quad \Gamma \end{cases}$$

The principle

Translate to a minimization problem

$$\min_{\mathbf{u}_{h,p} \in X_{h,p}} \mathcal{I}(\mathbf{u}_{h,p}; f, h) \equiv \frac{1}{2} \left(\|\mathcal{L}\mathbf{u}_{h,p} - f\|^2_{X_h,\Omega} + \|\mathcal{R}\mathbf{u}_{h,p} - h\|^2_{X_h,\Omega} \right)$$

Which reduces to:

$$\left(\mathcal{L}\mathbf{u}_{h,p}, \mathcal{L}\mathbf{v}_{h,p}\right)_\Omega + \left(\mathcal{R}\mathbf{u}_{h,p}, \mathcal{R}\mathbf{v}_{h,p}\right)_\Gamma = \left(f, \mathcal{L}\mathbf{v}_{h,p}\right)_\Omega + \left(h, \mathcal{R}\mathbf{v}_{h,p}\right)_\Gamma$$

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And finally to an algebraic system

$$\mathbf{A}\mathbf{u}_{h,p} = \mathbf{b}$$

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The finite dimensional spaces: C^0 nodal elements

All physical quantities represented by similar spaces

$$\phi(x,y) \rightarrow \phi_h(x,y) = \sum_{i,j} \phi_{i,j} h_i^p(x) h_j^p(y)$$

$$\mathbf{u}(x,y) \rightarrow \mathbf{u}_h(x,y) = \begin{bmatrix} \sum_{m,n} u^x_{m,n} h_m^p(x) h_n^p(y) \\ \sum_{k,l} u^y_{m,n} h_k^p(x) h_l^p(y) \end{bmatrix}$$

That is:

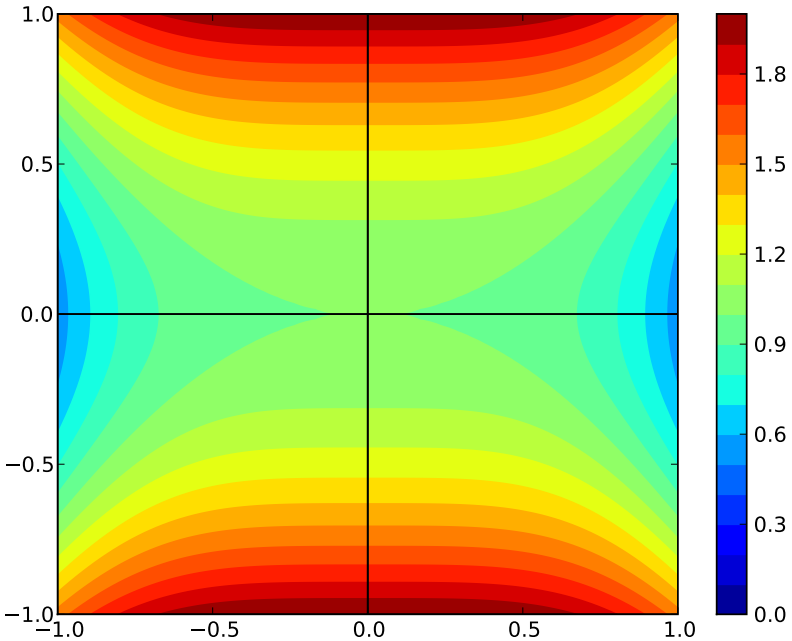
$$\phi_h(x,y) \in \text{span}\left\{ h_i^p(x) h_j^p(y) \right\}, \quad i,j = 0,\ldots,p$$

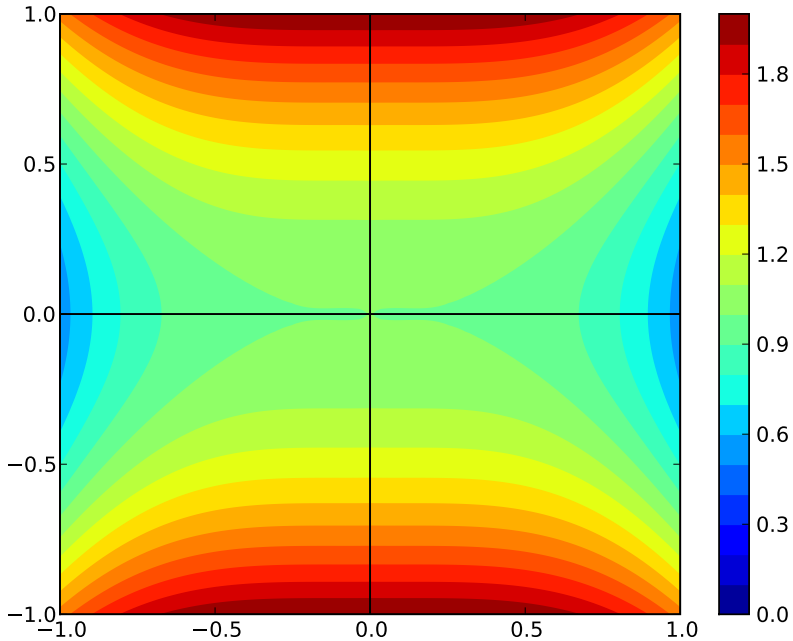
$$\mathbf{u}_h(x,y) \in \text{span}\left\{ h_m^p(x) h_n^p(y) \otimes h_k^p(x) h_l^p(y) \right\}, \quad m,n,k,l = 1,\ldots,p$$

$h_i^p(\xi)$ Lagrange interpolants over Gauss-Lobatto-Legendre points.

Numerical solution of 2D Poisson equation

$\phi(x, y)$ ▸ mimetic ϕ





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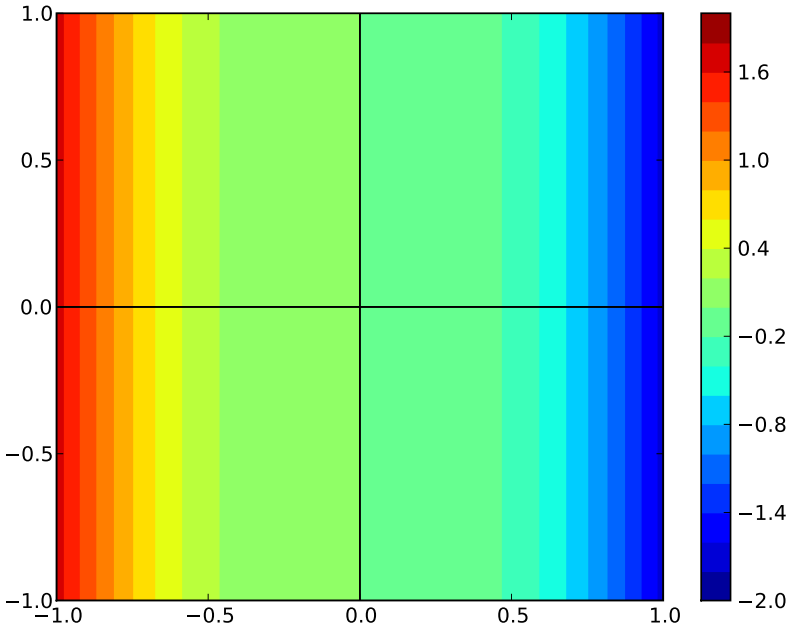
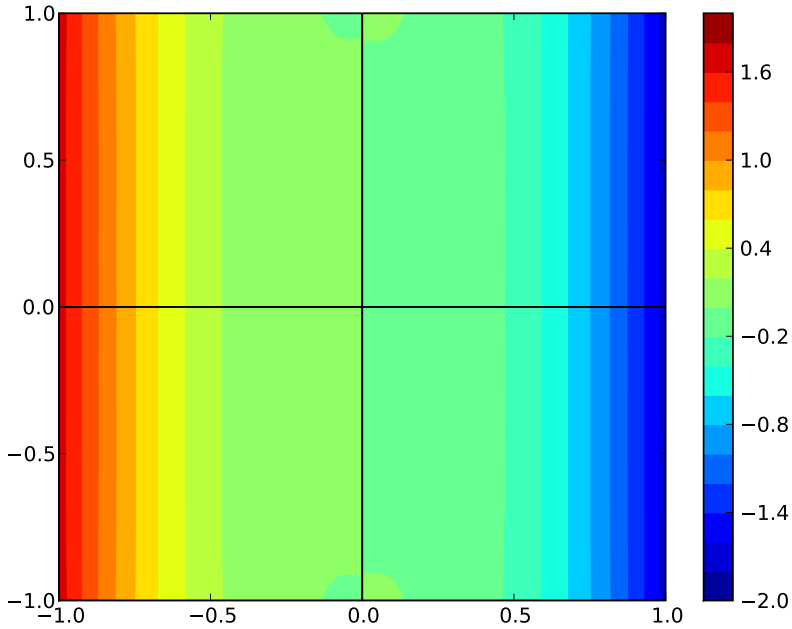
Numerical solution of 2D Poisson equation

$\mathbf{v}_x(x, y)$ ▸ mimetic $\mathbf{v}_x$ ▸ mimetic $\mathbf{q}_x$





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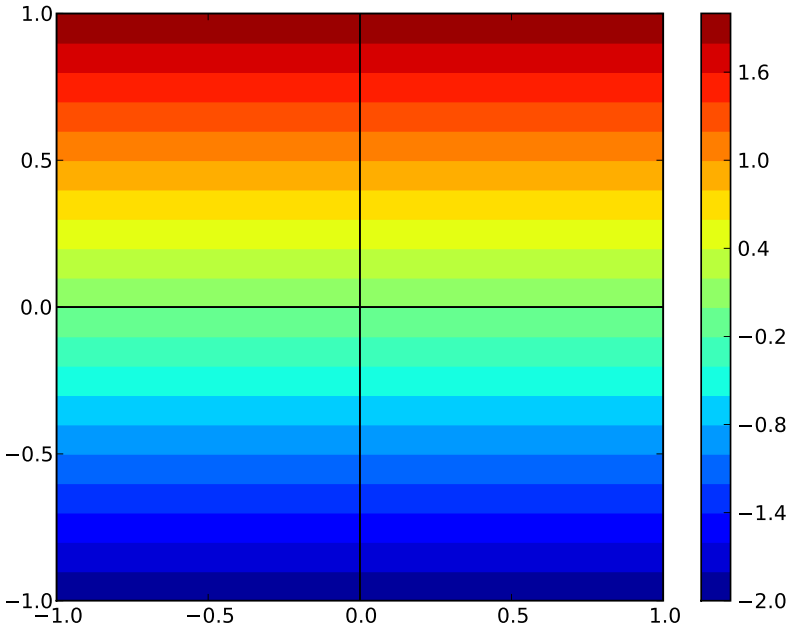
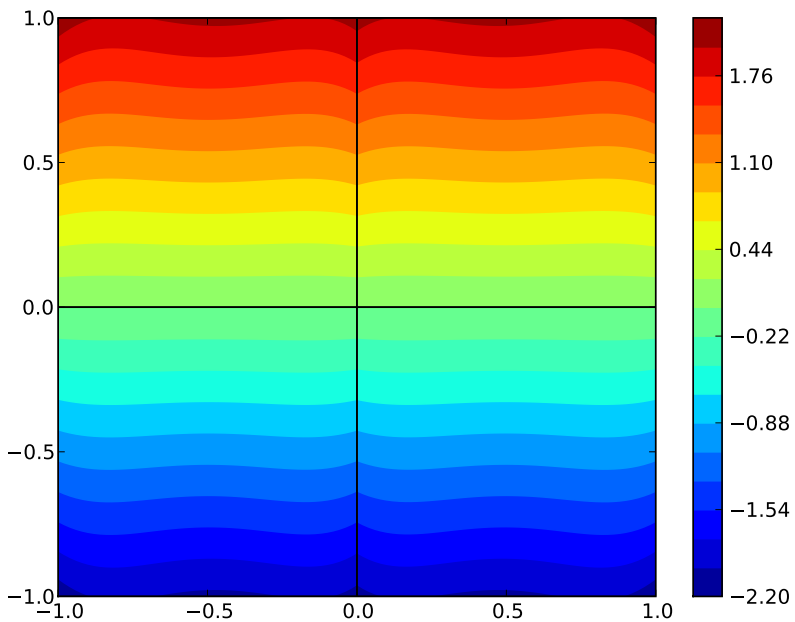
Numerical solution of 2D Poisson equation

$\mathbf{v}_y(x, y)$ ▸ mimetic $\mathbf{v}_y$ ▸ mimetic $\mathbf{q}_y$





Why it does not work?

We are not respecting the structure of the equations in the discrete setting

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Physical quantities and geometry

There is an intrinsic association between physical quantities and geometrical objects:

- Points: e.g. Electric potential, ϕ
- Lines: e.g. Electric field, $\mathbf{E}$, Magnetizing field, $\mathbf{H}$
- Surfaces: e.g. Magnetic flux, $\mathbf{B}$, Electric displacement field, $\mathbf{D}$
- Volumes: e.g. Charge density, ρ

These associations are intrinsic to the differential equations that relate the physical quantities:

$$\left\{\begin{array}{rcl} \nabla\cdot\mathbf{D} &=& \rho \\ \nabla\cdot\mathbf{B} &=& 0 \\ \nabla\times\mathbf{E} &=& -\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\times\mathbf{H} &=& \mathbf{J}+\frac{\partial D}{\partial t} \\ \mathbf{D} &=& \epsilon\mathbf{E} \\ \mathbf{B} &=& \mu\mathbf{H} \end{array}\right. \iff \left\{\begin{array}{rcl} \oiint_{\partial V}\mathbf{D}\cdot\mathrm{d}\mathbf{A} &=& Q(V) \\ \oiint_{\partial V}\mathbf{B}\cdot\mathrm{d}\mathbf{A} &=& 0 \\ \oint_{\partial S}\mathbf{E}\cdot\mathrm{dl} &=& -\frac{\partial}{\partial t}\iint_{S}\mathbf{B}\cdot\mathrm{d}\mathbf{A} \\ \oint_{\partial S}\mathbf{H}\cdot\mathrm{dl} &=& \iint_{S}\mathbf{J}\cdot\mathrm{d}\mathbf{A}+\frac{\partial}{\partial t}\iint_{S}\mathbf{D}\cdot\mathrm{d}\mathbf{A} \\ \mathbf{D} &=& \epsilon\mathbf{E} \\ \mathbf{B} &=& \mu\mathbf{H} \end{array}\right.$$

Classification by orientation

Configuration variables: variables that give the configuration of a physical system. How the system is described. (electric potential V, electric field vector $\mathbf{E}$, velocity vector $\mathbf{v}$). Associated to inner oriented manifolds

Source variables: variables that describe the sources of the field or the forces acting on the system. (electric current $\mathbf{J}$, electric induction $\mathbf{D}$, mass flux $\mathbf{q}$). Associated to outer oriented manifolds

Energy variables: variables obtained as the product of a configuration variable with a source variable.

Inner and outer orientation of geometrical objects

	inner orientation	outer orientation	
P	Inner orientation of a point: a positive orientation is the one of sink (for historical reasons).	Outer orientation of a volume. A positive orientation is the one with outward normals.	$\tilde{V}$
L	Inner orientation of a line: it is the basic notion used to give a meaning to the orientations of all other geometrical elements.	Outer orientation of a surface: it is the inner orientation of the line crossing the surface.	$\tilde{S}$
S	Inner orientation of a surface: it is a compatible orientation of its edges, i.e. a direction to go along its boundary.	Outer orientation of a line: it is the inner orientation of a surface crossing the line.	$\tilde{L}$
V	Inner orientation of a volume: it is a compatible orientation of its faces. It is equivalent to the screw rule.	Outer orientation of a point: it is the inner orientation of the volume containing the point.	$\tilde{P}$

Classification of physical laws

Topological laws

Are characterized by the fact that their validity is independent of the nature of the medium under consideration. Connect configuration variables with configuration variables and source variables with source variables. Are independent of metric since they are intrinsically integral equations (global).

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \phi = \mathbf{u}, \quad \nabla \times \mathbf{E} = 0, \ldots$$

Constitutive laws

Are characterized by the fact that their validity depends on the nature of the medium under consideration. They describe the behaviour of a material. Connect configuration variables with source variables. Depend on the metric since they are intrinsically local in nature.

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \mathbf{q} = \rho \mathbf{v}, \ldots$$

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Vector calculus obscures

Example: 2D Poisson equation for potential flow

$$\begin{cases} \nabla\phi &= \mathbf{v} \\ \nabla\cdot\mathbf{q} &= f \\ \mathbf{q} &= \rho\mathbf{v} \end{cases}$$

- There is no reference to which geometrical object the physical quantities are associated.
- There is no reference to inner or outer orientation.
- All this is given *a posteriori*. Right hand rule and so on.

How to solve this?

Vector calculus obscures

Example: 2D Poisson equation for potential flow

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How to solve this?

We need a proper framework!

Differential geometry

The Holy Grail: Differential Geometry

Differential Geometry: a refresher

We need to introduce 1 object and 4 operators:

- k-differential form or k-form
- wedge product, $\wedge$
- inner product, $(\cdot, \cdot)$
- exterior derivative, d
- Hodge-$\star$ operator, $\star$

k-forms

Mathematical definition

A k-differential form, or k-form, is a k-linear and antisymmetric mapping:

$$\omega^k : \underbrace{TM_{\mathbf{x}} \times \ldots \times TM_{\mathbf{x}}}_{k} \mapsto \mathcal{F}$$

Where $TM_{\mathbf{x}}$ is the tangent space of a k-differentiable manifold (smooth k-dimensional surface) at $\mathbf{x}$ and $\mathcal{F}$ is a field, which in this paper is $\mathbb{R}$.

What?

Can be seen as a machine that eats k-vectors and spits a number. Under integration can be seen as a machine that eats k-manifolds and returns a number. Intrinsically connected to geometry and integration.

Wedge product, $\wedge$

Mathematical definition

The wedge product, $\wedge$, in a n-dimensional space, is defined as a mapping:

$$\wedge : \Lambda^k \times \Lambda^l \mapsto \Lambda^{k+l}, \qquad k + l < n,$$

with the property that:

$$\omega^k \wedge \alpha^l = (-1)^{kl} \alpha^l \wedge \omega^k,$$

where Λ^k is the space of k-forms.

What?

Can be seen as a machine that eats a k-form and a l-form and returns a $(k + l)$-form.
Or simply as the generalization of the exterior product to arbitrary dimensions.

Proxies of k-forms

This means that, in $\mathbb{R}^n$, one can write any k-form as:

$$\omega^k = \sum_{i_1<\ldots<i_k} w_{i_1,\ldots,i_k}(\mathbf{x})\mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k},$$

where $\mathrm{d}x_i$ are the orthonormal basis 1-forms. In this way one can define the set of functions $\left\{w_{i_1,\ldots,i_k}(\mathbf{x})\right\}$ as proxies of k-forms, in the sense that there is a one-to-one correspondence between these sets and k-forms. Moreover, one can verify that these sets, in $\mathbb{R}^2$ for example, have a correspondence to well known entities: scalar fields as proxies for 0-forms and 2-forms, vector fields as proxies for 1-forms.

Inner product, $(\cdot, \cdot)$

Mathematical definition

The well known inner product in $\mathbb{R}^n$ and the correspondence between k-forms and their proxies induces the definition of an inner product for k-forms as a mapping $(\cdot, \cdot) : \Lambda^k \times \Lambda^k \mapsto \mathbb{R}$:

$$(\alpha^k, \beta^k) = \sum_{i_1 < \ldots < i_k} a_{i_1, \ldots, i_k}(\mathbf{x}) b_{i_1, \ldots, i_k}(\mathbf{x})$$

An L^2 inner product of k-forms, in a n-dimensional space, can also be defined as a mapping:

$$\left(\alpha^k, \beta^k\right)_\Omega = \int_\Omega \left(\alpha^k, \beta^k\right) \omega^n$$

Where we have introduced the volume n-form ω^n, on an n-dimensional space.

Exterior derivative, d

Mathematical definition

The exterior derivative d, in a n-dimensional space, is a mapping:

$$\mathrm{d} : \Lambda^k \mapsto \Lambda^{k+1}, \qquad k = 0, 1, \ldots, n-1,$$

which satisfies:

$$\mathrm{d}\left(\omega^k \wedge \alpha^l\right) = \mathrm{d}\omega^k \wedge \alpha^l + (-1)^k \omega^k \wedge \mathrm{d}\alpha^l, \qquad k + l < n$$

and

$$\mathrm{d}\mathrm{d}\omega^k = 0, \qquad \forall \omega \in \Lambda^k, \quad k < n-1$$

What?

It is simply a generalization of the well known differential operators, ∇, $\nabla\times$ and $\nabla\cdot$, to higher dimensions. In $\mathbb{R}^3$:

$$\mathrm{d}\phi^0 = \nabla\phi, \quad \mathrm{d}v^1 = \nabla \times \mathbf{v}, \quad \mathrm{d}e^2 = \nabla \cdot$$

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Hodge-$\star$ operator, $\star$

Mathematical definition

The Hodge-$\star$ operator, in an n-dimensional space, is a mapping:

$$\star : \Lambda^k \mapsto \Lambda^{n-k}, \qquad k \leq n,$$

that satisfies:

$$\alpha \wedge \star\beta = (\alpha, \beta)\, \omega^n, \qquad \forall \alpha, \beta \in \Lambda^k.$$

What?

A machine that transforms k-forms into twisted $(n-k)$-forms. Changes orientation from inner to outer and vice-versa.

So what do we get from this?

Elegance and compactness

Maxwell equations:

$$\mathrm{d}F^2 = 0, \quad \mathrm{d}G^2 = J^3, \quad \mathrm{d}J^3 = 0, \quad G^2 = \star F^2$$

Fundamental theorems:

$$\int_{\Omega_{k+1}} \mathrm{d}\omega^k = \int_{\partial\Omega_{k+1}} \omega^k$$

Clear relation and separation of objects

Connection between k-forms and $k+1$-manifolds:

$$\left[\mathrm{d}\omega^k, \Omega_{k+1}\right] = \left[\omega^k, \partial\Omega_{k+1}\right]$$

Constitutive equations are now expressed with the Hodge-$\star$ operator:

$$d^2 = \star_\epsilon e^1, \quad q^2 = \star_\rho v^1, \quad \ldots$$

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The de Rham complex

Additionally, one verifies that in sufficiently regular regions Ω the spaces of differential forms together with the exterior derivative d constitute an exact sequence, called the de Rham complex, which, in 3D is:

$$\begin{array}{ccccccccccc}
\mathbb{R} & \longrightarrow & \Lambda^0 & \xrightarrow{\mathrm{d}} & \Lambda^1 & \xrightarrow{\mathrm{d}} & \Lambda^2 & \xrightarrow{\mathrm{d}} & \Lambda^3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \star} & & \downarrow{\scriptstyle \star} & & \downarrow{\scriptstyle \star} & & \downarrow{\scriptstyle \star} & & \\
0 & \longleftarrow & \tilde{\Lambda}^3 & \xleftarrow[\mathrm{d}]{} & \tilde{\Lambda}^2 & \xleftarrow[\mathrm{d}]{} & \tilde{\Lambda}^1 & \xleftarrow[\mathrm{d}]{} & \tilde{\Lambda}^0 & \longleftarrow & \mathbb{R}
\end{array}$$

The de Rham complex

And in 2D reduces to:

$$\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & \Lambda^0 & \xrightarrow{\mathrm{d}} & \Lambda^1 & \xrightarrow{\mathrm{d}} & \Lambda^2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \star} & & \downarrow{\scriptstyle \star} & & \downarrow{\scriptstyle \star} & & \\
0 & \longleftarrow & \tilde{\Lambda}^2 & \xleftarrow[\mathrm{d}]{} & \tilde{\Lambda}^1 & \xleftarrow[\mathrm{d}]{} & \tilde{\Lambda}^0 & \longleftarrow & \mathbb{R}
\end{array}$$

Which is equivalent to the more familiar: ▸ more

$$\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & H^1 & \xrightarrow{\nabla} & H^1(\mathrm{curl}) & \xrightarrow{\nabla\times} & L^2 & & \\
 & & \downarrow{\scriptstyle \star} & & \downarrow{\scriptstyle \star} & & \downarrow{\scriptstyle \star} & & \\
 & & L^2 & \xleftarrow[\nabla\cdot]{} & H^1(\mathrm{div}) & \xleftarrow[\nabla^\perp]{} & H^1 & \longleftarrow & \mathbb{R}
\end{array}$$

The Tonti diagram: 2D Poisson equation

The equation

$$\left\{\begin{array}{rcl} \nabla\phi & = & \mathbf{v} \\ \nabla\cdot\mathbf{q} & = & f \\ \mathbf{q} & = & \rho\mathbf{v} \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} \mathrm{d}\phi^0 & = & v^1 \\ \mathrm{d}q^1 & = & f^2 \\ q^1 & = & \star_\rho v^1 \end{array}\right.$$

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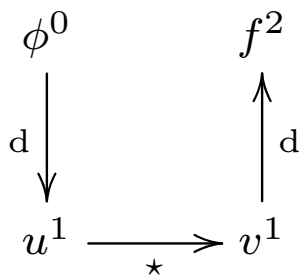
The diagram



Numerical solution: discretizations

Dual grid methods

- Depends on the existence of a pair of topologically dual grids
- One-to-one correspondence between dual variables
- Simple Hodge-$\star$ operator
- All equations satisfied globally

Elimination methods

- Sacrifices one of the equillibrium equations
- Satisfies exactly the other equilibrium equation and the constitutive equation locally

Primal-dual grid methods

- Satisfies exactly the equilibrium equations
- Relaxes the constitutive equation, being enforced weakly

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Weak material laws: the role of least-squares

The idea

- Impose the constitutive equation weakly
- Hodge-$\star$ operator defined implicitly
- Minimize local discrepancy between dual variables

The implementation

$$\text{Seek } (\phi_h^0, v_h^1, q_h^1) \text{ in } \Lambda_h^0 \times \Lambda_h^1 \times \Lambda_h^1 \text{ such that} \tag{1}$$
$$\mathcal{I}(\phi_h^0, v_h^1, q_h^1) = \tfrac{1}{2}\left(\| \star q_h^1 + v_h^1 \|_0^2 + \| \mathrm{d} q_h^1 - f^2 \|_0^2 \right)$$

$$\text{subject to:} \quad \mathrm{d}\phi_h^0 = v_h^1$$

Weak material laws: the role of least-squares

If the subspaces Λ_h^0, Λ_h^1 and Λ_h^2 are chosen in such a way that they constitute a de Rham complex:

$$\mathbb{R} \to \Lambda_h^0 \overset{\mathrm{d}}{\longmapsto} \Lambda_h^1 \overset{\mathrm{d}}{\longmapsto} \Lambda_h^2 \mapsto 0$$

then $\mathrm{d}\phi_h^0 = v_h^1$ is satisfied exactly. The problem becomes:

$$\text{Seek } (\phi_h^0, q_h^1) \text{ in } \Lambda_h^0 \times \Lambda_h^1 \text{ such that} \tag{2}$$
$$\mathcal{I}(\phi_h^0, q_h^1) = \tfrac{1}{2}\left(\| \star q_h^1 + \mathrm{d}\phi^0\|_0^2 + \|\mathrm{d}q_h^1 - f^2\|_0^2\right)$$

In this way, the Hodge-$\star$ operator is implemented as L^2 projections between the different dual spaces.

Application to the 2D Poisson equation

Find adequate subspaces Λ_h^0, Λ_h^1 and Λ_h^2 must be specified. Since one will use a spectral/hp LS method, these spaces are defined as:

$$\Lambda_{h,p}^0 = \text{span}\left\{h_i^p(x)h_j^p(y)\right\} \quad , \quad i = 0,\dots,p \quad j = 0,\dots,p$$

$$\Lambda_{h,p}^1 = \text{span}\left\{\tilde{h}_i^{p-1}(x)h_j^p(y) \otimes h_n^p(x)\tilde{h}_m^{p-1}(y)\right\} \quad , \quad i,m = 1,\dots,p \quad j,n = 0,\dots,p$$

$$\Lambda_{h,p}^2 = \text{span}\left\{\tilde{h}_i^{p-1}(x)\tilde{h}_j^{p-1}(y)\right\} \quad , \quad i = 1,\dots,p \quad j = 1,\dots,p$$

- $h_i^p(\xi)$: i-th Lagrange interpolant of order p throught Gauss-Lobatto-Legendre points
- $\tilde{h}_i^p(\xi)$: i-th Lagrange interpolant of order p throught Gauss points
- Degrees of freedom are located where they should be: at nodal points (for 0-forms), at edges (for 1-forms) and at volumes (for 2-forms).
- Different continuity properties
- These subspaces constitute a de Rham complex

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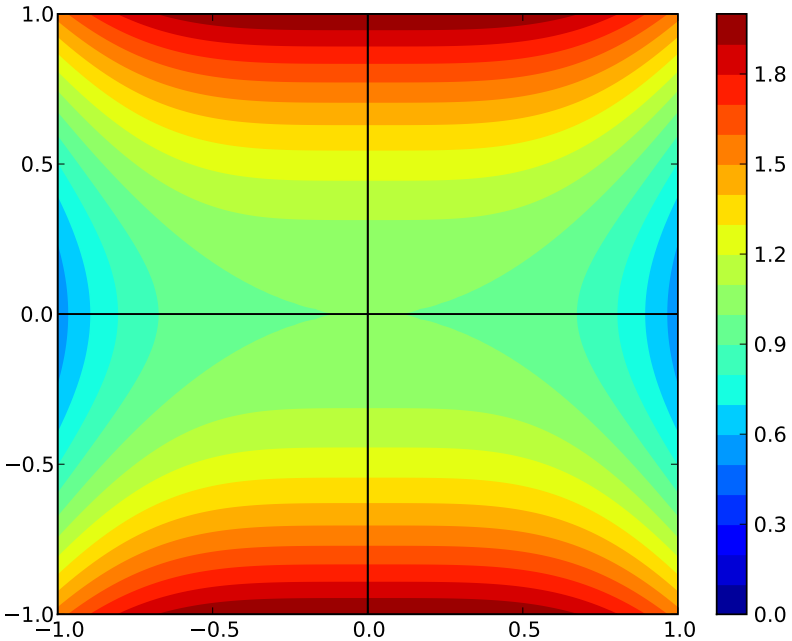
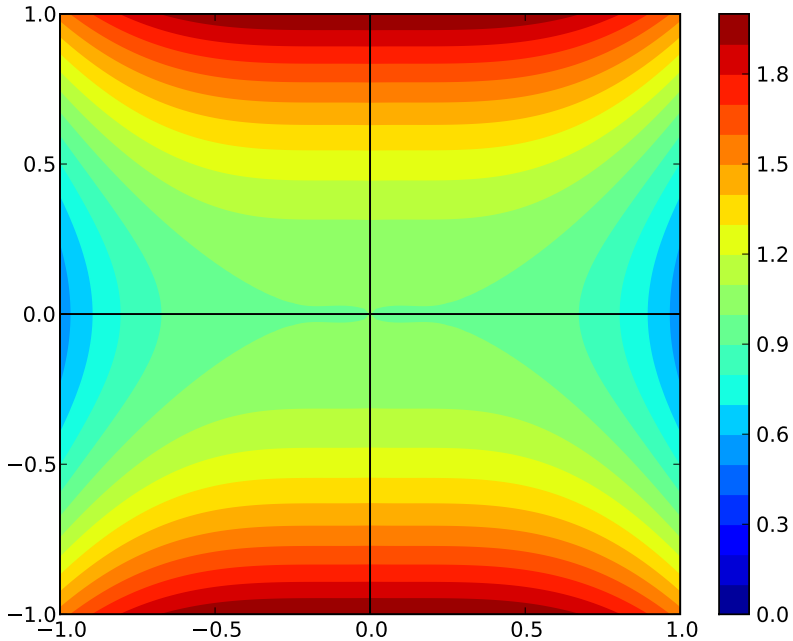
Numerical results

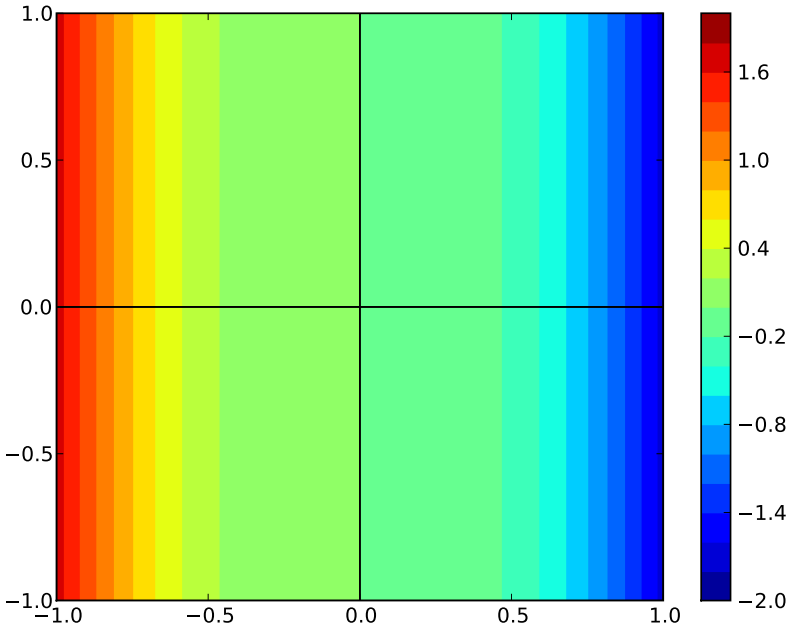
$\phi(x, y)$ ▸ standard ϕ

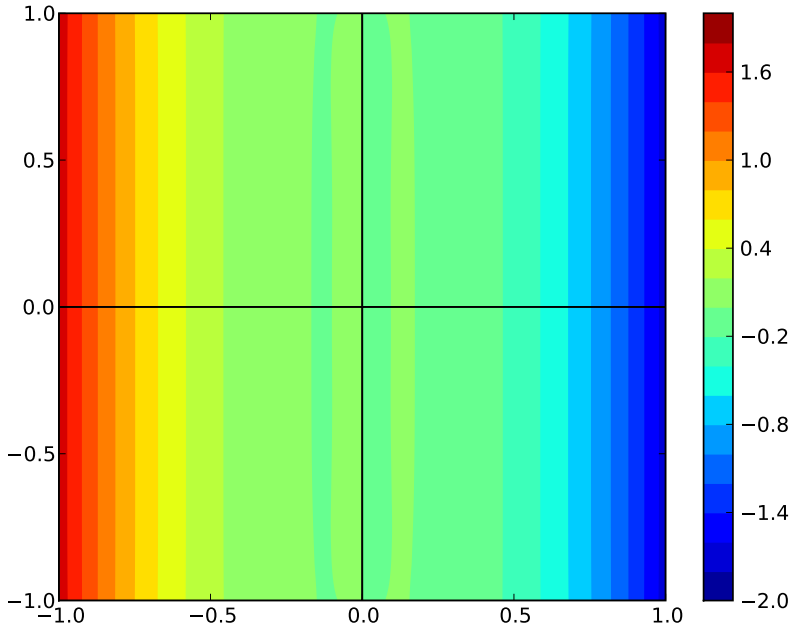




Numerical results

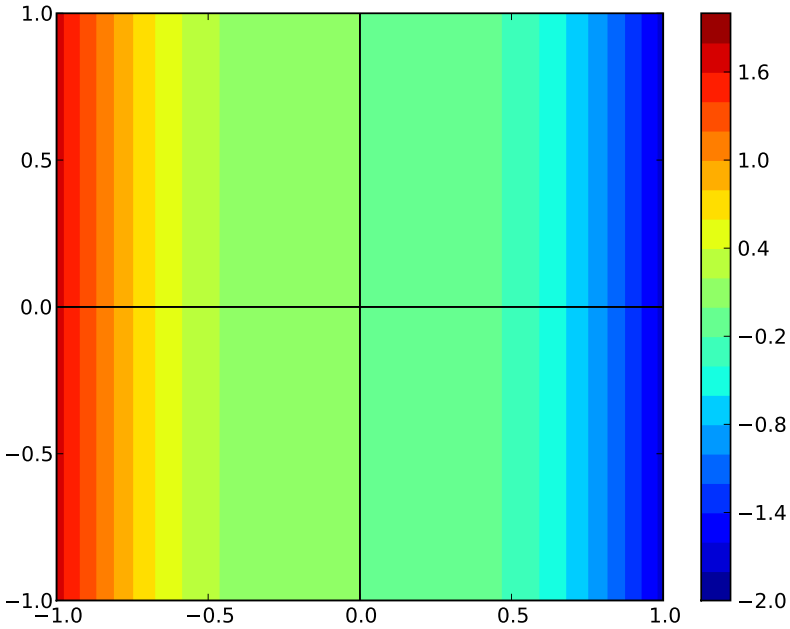
$\mathbf{v}_x(x, y)$ ▸ standard $\mathbf{v}_x$

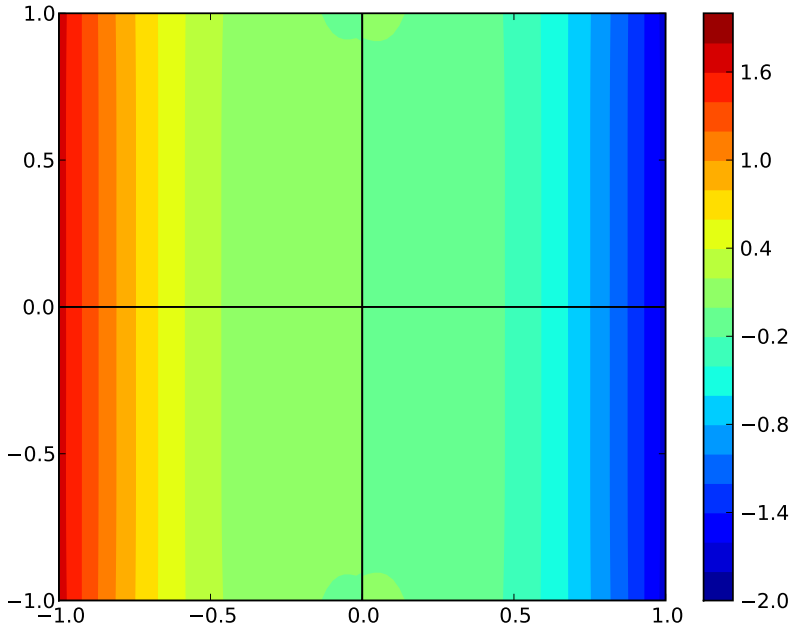




Numerical results

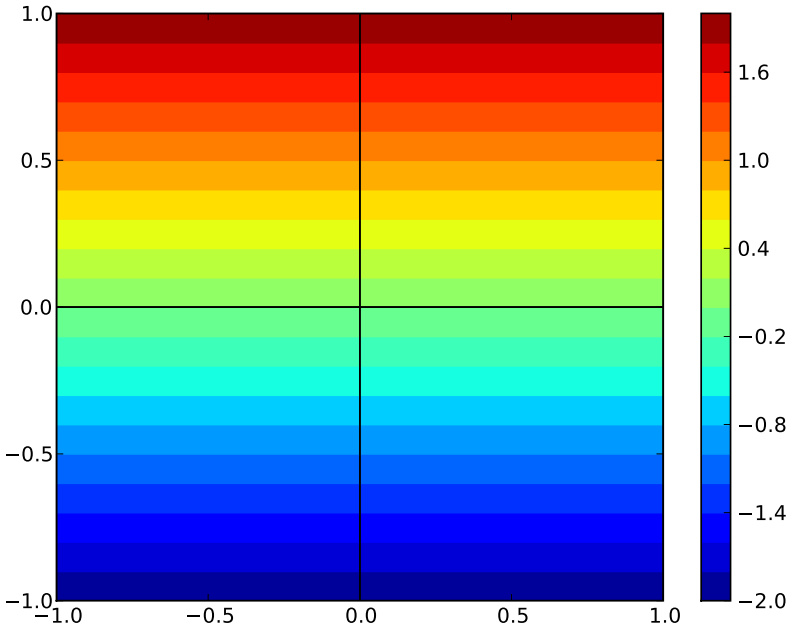
$\mathbf{q}_x(x, y)$ ▸ standard $\mathbf{v}_x$

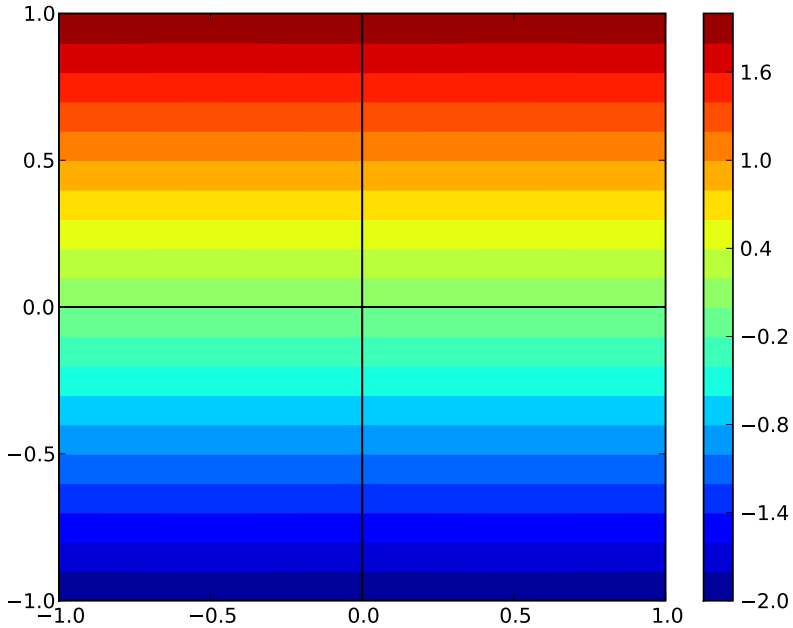




Numerical results

$\mathbf{v}_y(x, y)$ ▸ standard $\mathbf{v}_y$





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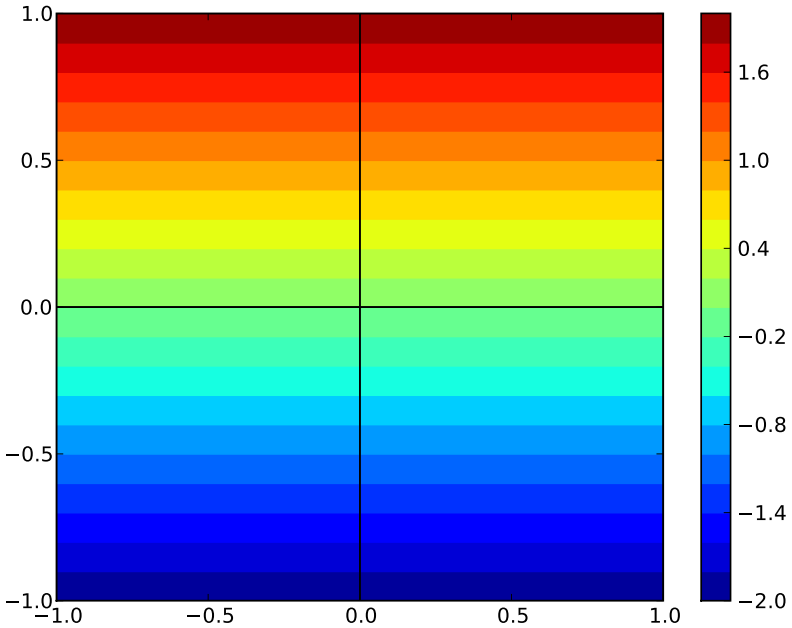
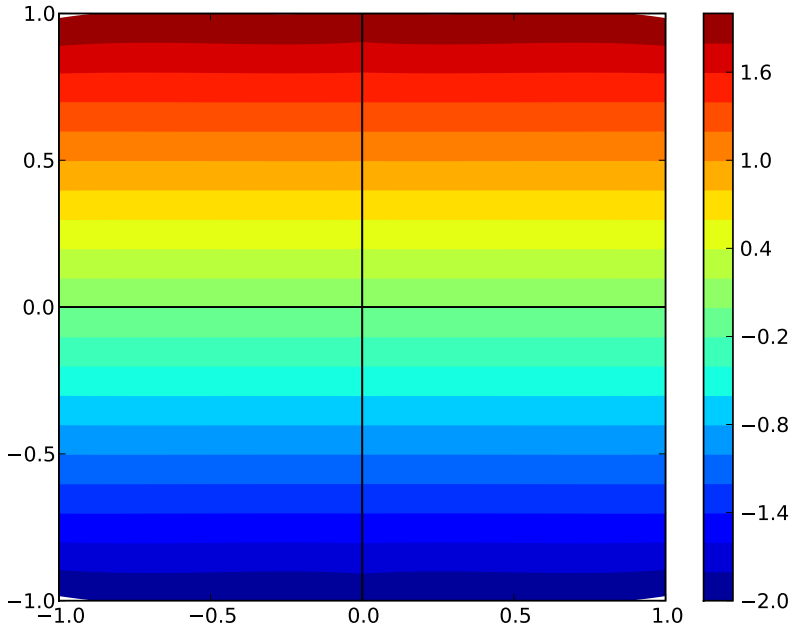
Numerical results

$\mathfrak{q}_y(x, y)$ ▸ standard $\mathbf{v}_y$





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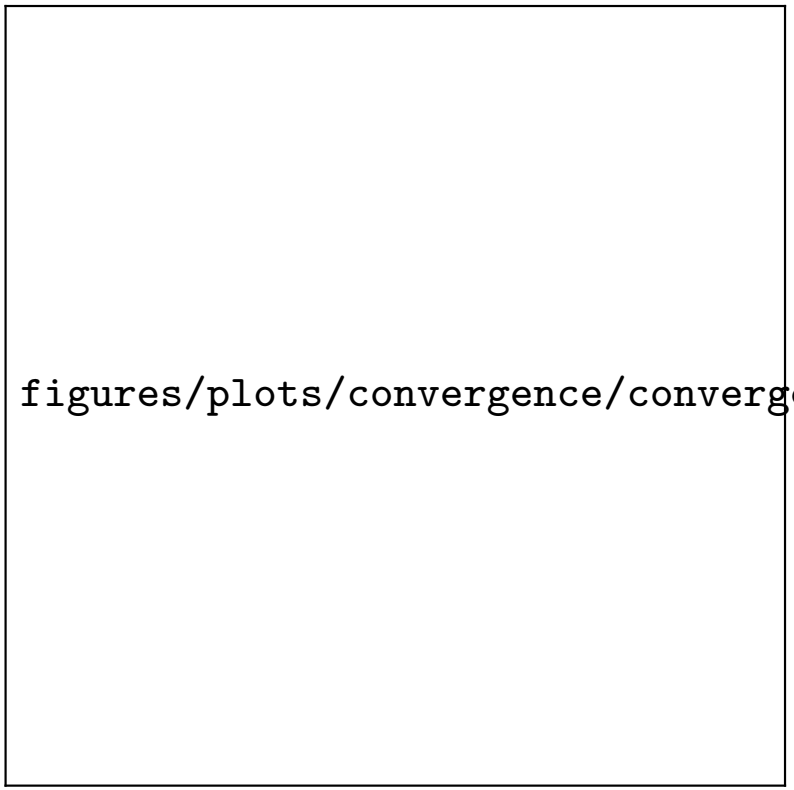
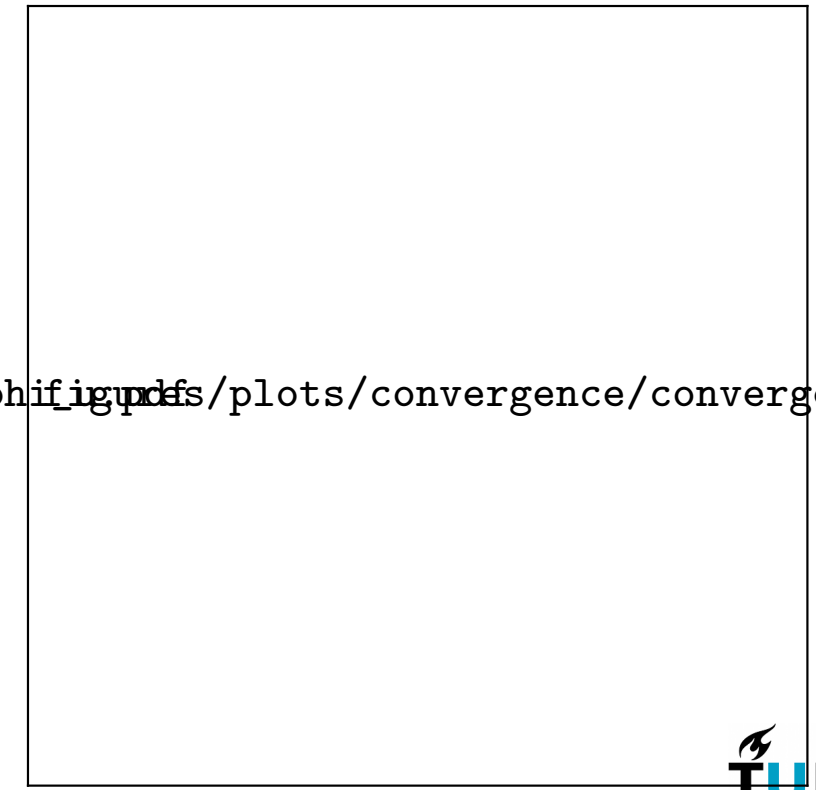
Numerical results

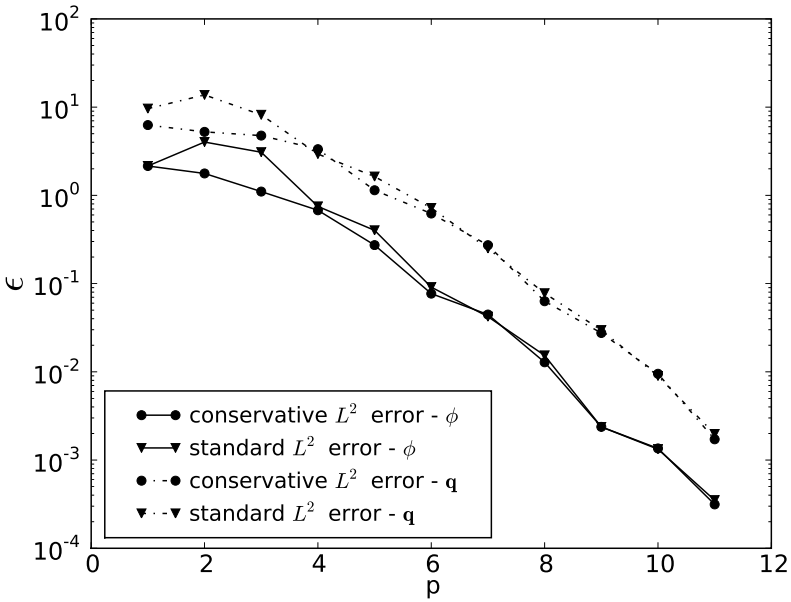
Convergence results

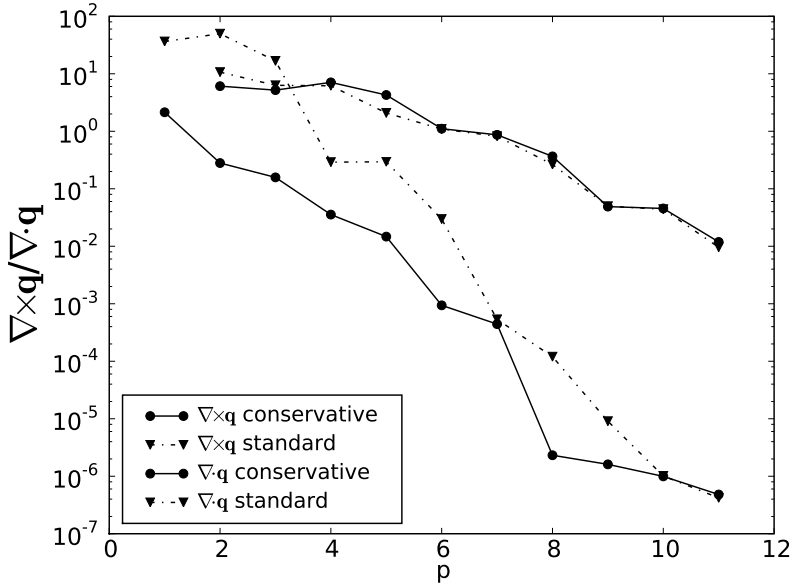




Numerical results: Histopolant basis functions

Convergence results





For Histopolants see: Robidoux, *Polynomial Histopolation, Superconvergent Degrees Of Freedom, And Pseudospectral Discrete Hodge Operators*, to appear.

Summary

- Physical quantities are inherently geometrical
- Structure of PDE's must be obeyed
- There is more to life than scalars and vectors

Future work

- Curved domains
- 3 dimensions
- 4 dimensions: space-time

The end!

The end!

Further reading

Tonti, E.: *On the formal structure of physical theories*. Consiglio Nazionale delle Ricerche, Milano (1975)

Bochev, P. and Hyman, J.: Principles of mimetic discretizations of differential operators. IMA 142, 89–119 (2006)

Mattiussi, C.: An analysis of finite volume, finite element, and finite difference methods using some concepts from algebraic topology. J. Comp. Physics 133, 289–309 (1997)

Desbrun, M. and Kanso, E. and Tong, Y.: Discrete differential forms for computational modeling. SIGGRAPH '05: ACM SIGGRAPH 2005 Courses (2005)

Bossavit, A.: On the geometry of electromagnetism. J. Japan Soc. Appl. Electromagn. & Mech. 6 (1998)

The de Rham complex in 2D explained

Usually the 2-dimensional case is viewed as a special case and hence it is expressed by two exact sequences:

$$\mathbb{R} \hookrightarrow H^1 \xrightarrow{\nabla} H^1(\mathrm{curl}) \xrightarrow{\nabla\times} L^2 \tag{3}$$
$$\mathbb{R} \hookrightarrow H^1 \xrightarrow{\nabla^\perp} H^1(\mathrm{div}) \xrightarrow{\nabla\cdot} L^2$$

with

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{e}_x + \frac{\partial\phi}{\partial y}\mathbf{e}_y \tag{4}$$
$$\nabla^\perp\phi = -\frac{\partial\phi}{\partial y}\mathbf{e}_x + \frac{\partial\phi}{\partial x}\mathbf{e}_y \tag{5}$$
$$\nabla\times\mathbf{W} = \frac{\partial\mathbf{W}_y}{\partial x}\mathbf{e}_x - \frac{\partial\mathbf{W}_x}{\partial y}\mathbf{e}_y \tag{6}$$
$$\nabla\cdot\mathbf{W} = \frac{\partial\mathbf{W}_x}{\partial x} + \frac{\partial\mathbf{W}_y}{\partial y}$$

back

The de Rham complex in 2D explained

These two exact sequences are obtained by a restriction of the 3D exact sequence:

$$\mathbb{R} \hookrightarrow H^1 \xrightarrow{\nabla} H^1(\mathrm{curl}) \xrightarrow{\nabla\times} H^1(\mathrm{div}) \xrightarrow{\nabla\cdot} L^2$$

to a planar 2D surface embedded in $\mathbb{R}^3$, for example the xy-plane. This, in turn, reduces to the pair of exact sequences Eq. (3) and Eq. (**??**), since the vectors of the form $\phi \mathbf{e}_z$ can be identified with scalar functions. The odd operator $\nabla^\perp$ is, then, nothing but the result of applying the 3D $\nabla\times$ to $\phi \mathbf{e}_z$.
The full de Rham complex in 2D becomes:

$$\begin{array}{ccccccc}
\mathbb{R} & \longrightarrow & H^1 & \xrightarrow{\nabla} & H^1(\mathrm{curl}) & \xrightarrow{\nabla\times} & L^2 \\
 & & \downarrow{\scriptstyle\star} & & \downarrow{\scriptstyle\star} & & \downarrow{\scriptstyle\star} \\
 & & L^2 & \xleftarrow[\nabla\cdot]{} & H^1(\mathrm{div}) & \xleftarrow[\nabla^\perp]{} & H^1 \longleftarrow \mathbb{R}
\end{array}$$

back

The de Rham complex in 2D explained

It is important to realize that this full exact complex is always relative to the proxies of the differential forms, that is, scalar and vector fields, not the differential k-forms to which they are associated. This is the important point, since this is the reason why both De Rham complexes are equivalent.
Let us therefore show how the differential formulation agrees with the vector formulation with its special characteristics in 2D:

$$\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & \Lambda^0 & \xrightarrow{\mathrm{d}} & \Lambda^1 & \xrightarrow{\mathrm{d}} & \Lambda^2 & & \\
 & & \downarrow \star & & \downarrow \star & & \downarrow \star & & \\
 & & \tilde{\Lambda}^2 & \xleftarrow[\mathrm{d}]{} & \tilde{\Lambda}^1 & \xleftarrow[\mathrm{d}]{} & \tilde{\Lambda}^0 & \longleftarrow & \mathbb{R}
\end{array}$$

Where $\tilde{\Lambda}^0$, $\tilde{\Lambda}^1$ and $\tilde{\Lambda}^2$ are the spaces of twisted forms (as in Burke (1985) section 28, or Bossavit (1998) Japanese papers chapter *(2):Geometrical objects*), and the untilded ones are the spaces of forms.

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The de Rham complex in 2D explained

To show that both exact complexes are equivalent one must show how to pass from forms to proxies and from proxies to forms. This is done by using the sharp, $\sharp$, and flat, $\flat$, operators, respectively. The sharp, $\sharp$, and flat, $\flat$, acting on a 0-form (ϕ^0) and a scalar field (ϕ), give:

$$\left(\phi^0\right)^\sharp = \phi, \quad \phi^\flat = \phi^0$$

The sharp, $\sharp$, and flat, $\flat$, acting on a 1-form ($\alpha^1 = f\mathrm{dx} + g\mathrm{dy}$) and a vector field ($\mathbf{A} = f\mathbf{e}_x + g\mathbf{e}_y$), give:

$$\left(\alpha^1\right)^\sharp = f\mathbf{e}_x + g\mathbf{e}_y, \quad \mathbf{A}^\flat = f\mathrm{dx} + g\mathrm{dy}$$

The sharp, $\sharp$, and flat, $\flat$, acting on a twisted 1-form ($\tilde{\beta}^1 = -g\mathrm{dx} + f\mathrm{dy}$) and a vector field ($\mathbf{B} = f\mathbf{e}_x + g\mathbf{e}_y$):

$$\left(\tilde{\beta}^1\right)^\sharp = f\mathbf{e}_x + g\mathbf{e}_y, \quad \mathbf{B}^\flat = -g\mathrm{dx} + f\mathrm{dy}$$

back

The de Rham complex in 2D explained

The sharp, $\sharp$, and flat, $\flat$, acting on a 2-form ($\omega^2 = w\mathrm{dxdy}$) and a scalar field ($w$):

$$\left(\omega^2\right)^\sharp = w, \quad \mathbf{w}^\flat = w\mathrm{dxdy}$$

The case of twisted 0-forms and twisted 2-forms are identical to the corresponding standard forms.
For the 2D case, these operations can be summarized by

$$\star 1 = \mathrm{dxdy}, \quad \star \mathrm{dx} = \mathrm{dy}, \quad \star \mathrm{dy} = -\mathrm{dx} \quad \star\, dxdy = 1$$

Burke (1985) section 28, or Bossavit (1998) Japanese papers chapter *(2):Geometrical objects* See also, Burke (1985) section 28, Bossavit (1998) Japanese papers chapter *(2):Geometrical objects*, Marsden (2002) p.432 and Bossavit (2005), p. 21 and p.23.
We can see now that the special form of the 2D De Rham complex in vector form results from converting the usual De Rham complex for 2D in differential form to its vectorial representation using the above relations.
The top exact complexes are identical, on the proxies. On the 0-forms:

$$\left(\mathrm{d}\phi^0\right)^\sharp = \left(\frac{\partial \phi}{\partial x}\mathrm{dx} + \frac{\partial \phi}{\partial y}\mathrm{dy}\right)^\sharp = \frac{\partial \phi}{\partial x}\mathbf{e}_x + \frac{\partial \phi}{\partial y}\mathbf{e}_y = \nabla\phi$$

back

The de Rham complex in 2D explained

On the 1-forms:

$$(\mathrm{d}\alpha^1)^\sharp = \left[\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathrm{dxdy}\right]^\sharp = \frac{\partial f}{\partial x} - \frac{\partial g}{\partial y} = \nabla \times (\alpha^1)^\sharp = \nabla \times \mathbf{A}$$

Which is exactly the same. Now, the bottom exact complex in Eq. (**??**) is identical to the one Eq. (**??**), on the proxies. On the twisted 0-forms:

$$\left(\mathrm{d}\tilde{\phi}^0\right)^\sharp = \left(\frac{\partial \phi}{\partial x}\mathrm{dx} + \frac{\partial \phi}{\partial y}\mathrm{dy}\right)^\sharp = \frac{\partial \phi}{\partial y}\mathbf{e}_x - \frac{\partial \phi}{\partial x}\mathbf{e}_y = -\nabla^\perp\left(\tilde{\phi}^0\right)^\sharp$$

Remembering that the differential of a twisted form is a twisted form. On the twisted 1-forms:

$$\left(\mathrm{d}\tilde{\beta}^1\right)^\sharp = \left[\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)\mathrm{dxdy}\right]^\sharp = (\tilde{w}^2)^\sharp = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = \nabla \cdot \left(\tilde{\beta}^1\right)^\sharp$$

where we have used, as before, $\tilde{\beta}^1 = -g\mathrm{dx} + f\mathrm{dy}$.

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The de Rham complex in 2D explained

Summarizing: The 2D case is not special in differential geometry, but its representation in vector form is markedly different from the 3D case.

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