Numerical Solution of Advection Equation with the Discretization of the Lie Derivative

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Outline

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- Space-time approach to 1D advection

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- Where to define the degrees of freedom?
- Geometrical degrees of freedom
- Reconstruction of physical quantities

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- Vector calculus vs Differential geometry
- Discrete advection equation
- Results

Conclusions
- Conclusions and future work
- Acknowledgements

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1D time dependent advection

1D time dependent advection

Advective form:

$$\frac{\partial \overline{q}}{\partial t} + v\frac{\partial \overline{q}}{\partial x} = 0$$

Conservative form:

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}(qv) = 0$$

1D time dependent advection

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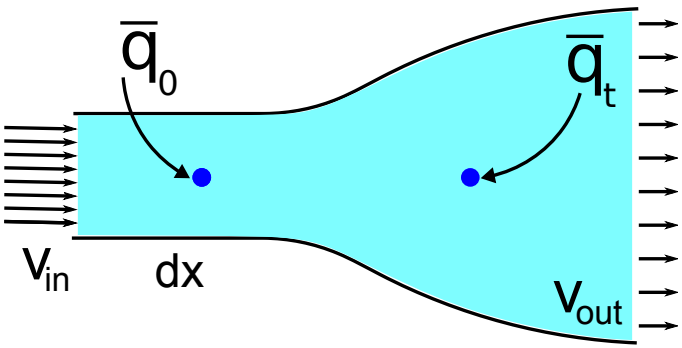

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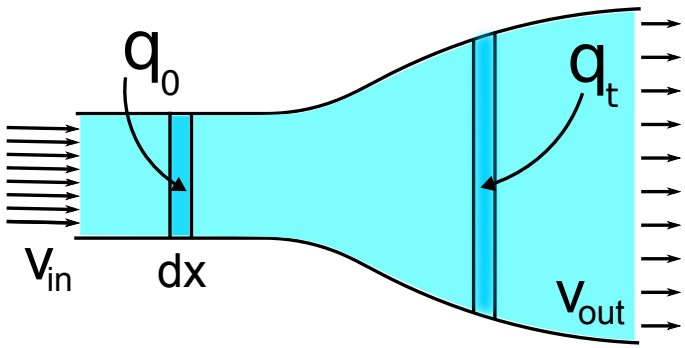

$$\sigma\overline{q} = q$$

1D time dependent advection

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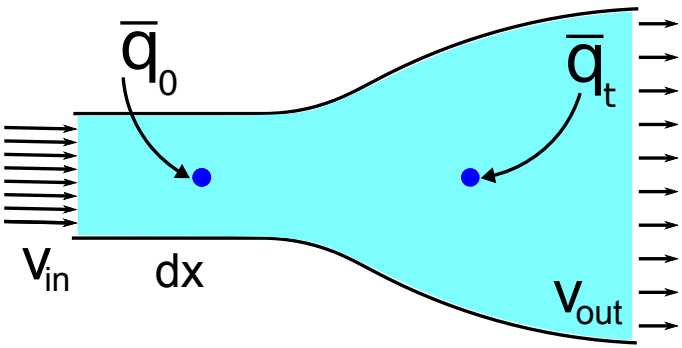

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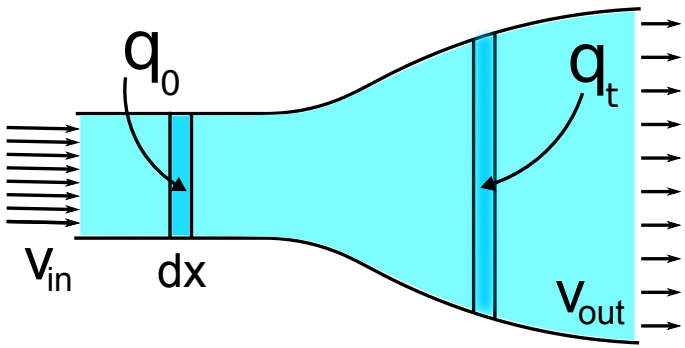

$$\sigma\overline{q} = q$$

Analytical solutions

$$\overline{q}(x,t) = \overline{q}_{t_0}(x - vt)$$

$$q(x,t) = \sigma(x)q_{t_0}(x - vt)$$

Space-time approach to 1D advection

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Conservative form:

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(qv\right) = 0$$

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Integrate $\downarrow$ in space-time

$$\iint_{\overline{\Omega}} \left[\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(qv\right)\right] \mathrm{d}x\mathrm{d}t = 0$$

Space-time approach to 1D advection

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Integrate $\downarrow$ in space-time

$$\iint_{\overline{\Omega}} \left[\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}(qv) \right] \mathrm{d}x\mathrm{d}t = 0$$

$$\overline{\nabla} = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial t}\vec{e}_t \quad \downarrow \quad \begin{array}{l} \vec{V} = v\vec{e}_x + \vec{e}_t \\ \vec{J} = q\vec{J} \end{array}$$

$$\iint_{\overline{\Omega}} \overline{\nabla} \cdot \vec{J} \, \mathrm{d}x\mathrm{d}t = 0$$

Space-time approach to 1D advection

$$\iint_{\overline{\Omega}} \overline{\nabla} \cdot \vec{J} \, \mathrm{d}x\mathrm{d}t = \int_{\partial\overline{\Omega}} \vec{J} \cdot \vec{n} \, \mathrm{d}s$$

Space-time approach to 1D advection

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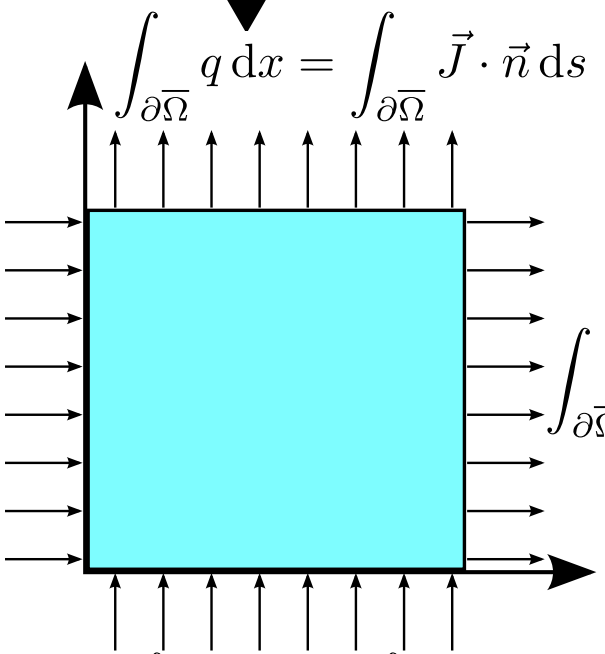

Where to define the degrees of freedom?

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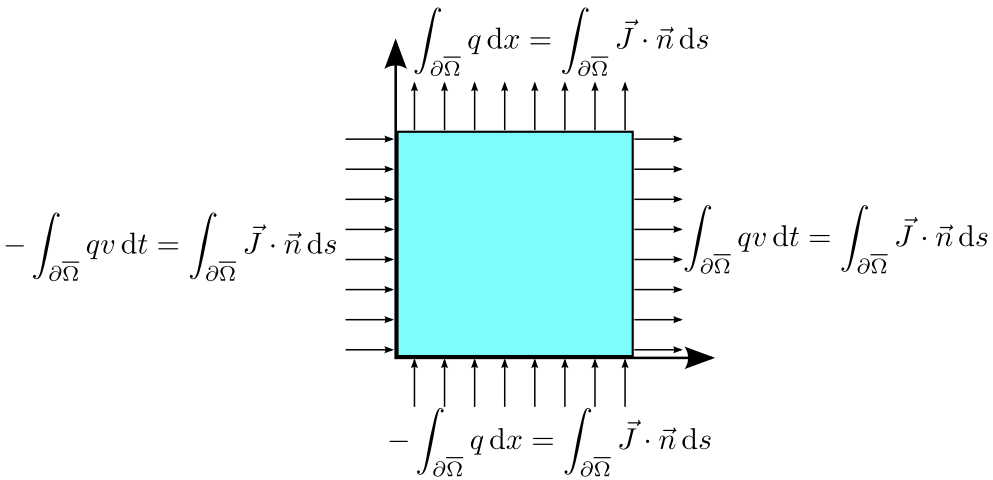


- How to represent $q(x,t)$? Which degrees of freedom to use?
- How to represent $\vec{J}$? Which degrees of freedom to use?
- How to define the relation (mapping) between $q(x,t)$ and $\vec{J}$?

$$\iota_{\vec{V}}\star : (q(x,t), \vec{V}) \mapsto \vec{J}(x,t)$$

Geometrical degrees of freedom

Geometrical degrees of freedom

Fundamental theorem of calculus

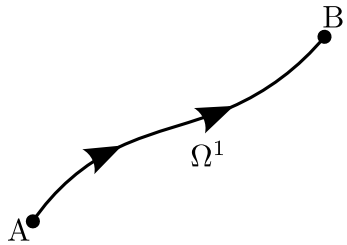
Kelvin-Stokes theorem

Gauss theorem

Geometrical degrees of freedom

Fundamental theorem of calculus

$$\int_{\Omega^1} \nabla\varphi \cdot \mathrm{d}\vec{s} = \varphi(\Omega^1_B) - \varphi(\Omega^1_A)$$



Kelvin-Stokes theorem

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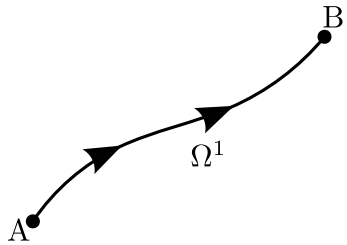
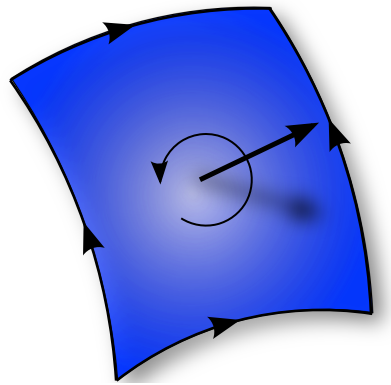


Kelvin-Stokes theorem

$$\int_{\Omega^2} (\nabla \times \vec{V}) \cdot \vec{n}\, \mathrm{d}S = \int_{\partial\Omega^2} \vec{V} \cdot \mathrm{d}\vec{s}$$



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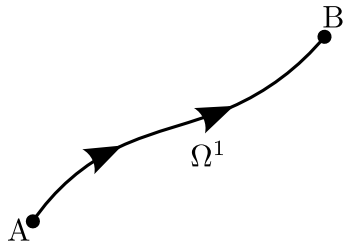

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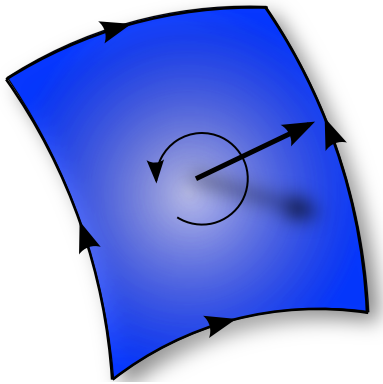
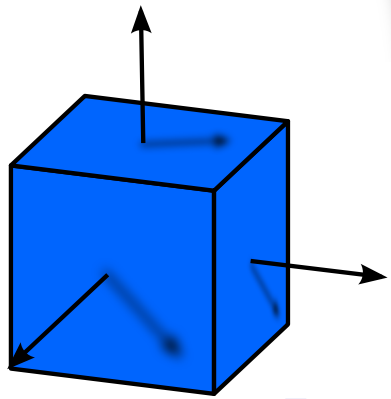
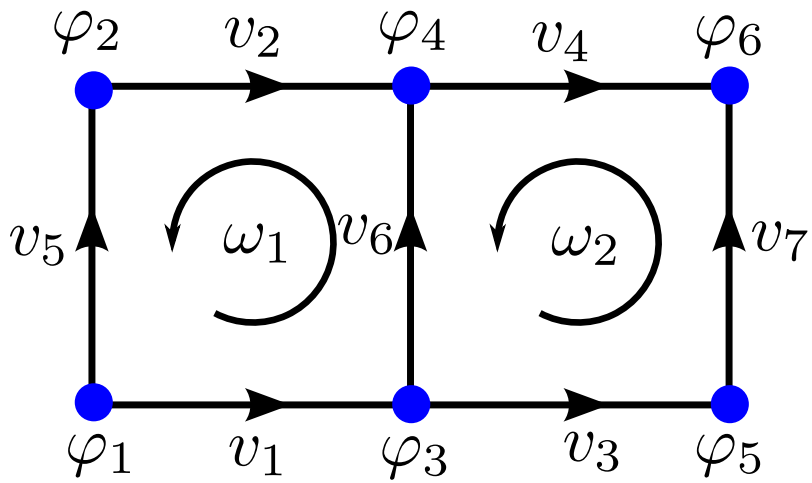


Gauss theorem

$$\int_{\Omega^3} (\nabla \cdot \vec{V}) \mathrm{d}V = \int_{\partial\Omega^3} \vec{V} \cdot \vec{n}\, \mathrm{d}S$$



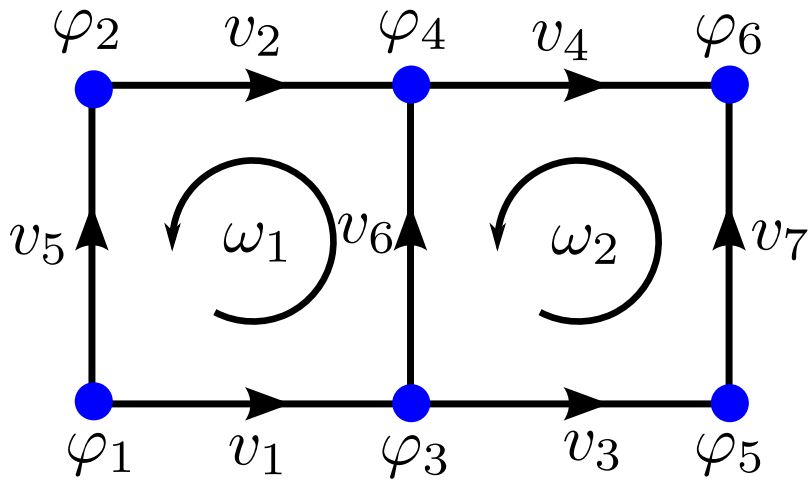
Differential operators



Differential operators

Gradient

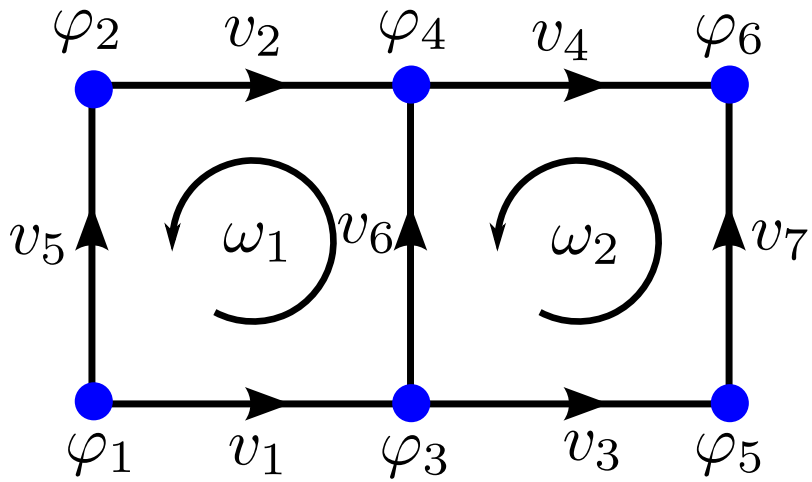
$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}$$



Differential operators

Curl

$$\begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 1 \end{bmatrix}$$



Reconstruction of physical quantities

Reconstruction of physical quantities

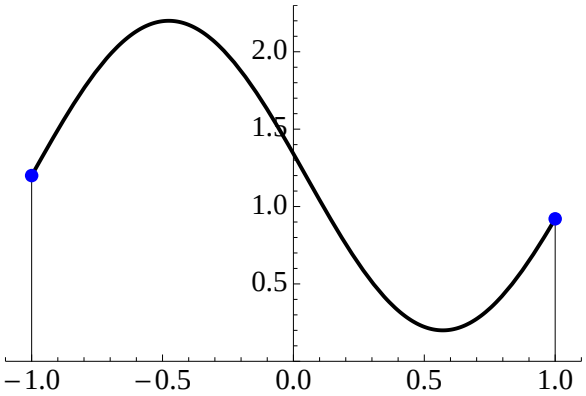
Nodal quantities: Lagrange interpolants

$$h_i^p(x_j) = \delta_{ij}$$

$i, j = 1, \ldots, p+1$

Reconstruction of physical quantities

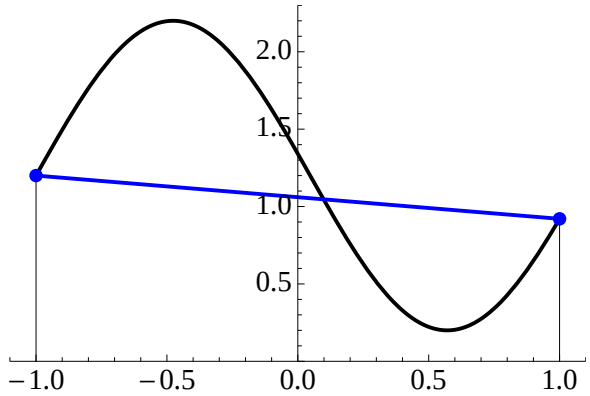
Nodal quantities: p=1



Reduce
(Discretize)
$\mathcal{R}$

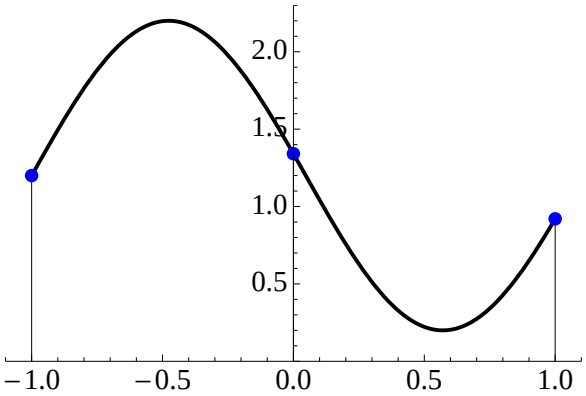
$$[\widetilde{\varphi}_i] = [\varphi(-1), \varphi(1)]$$

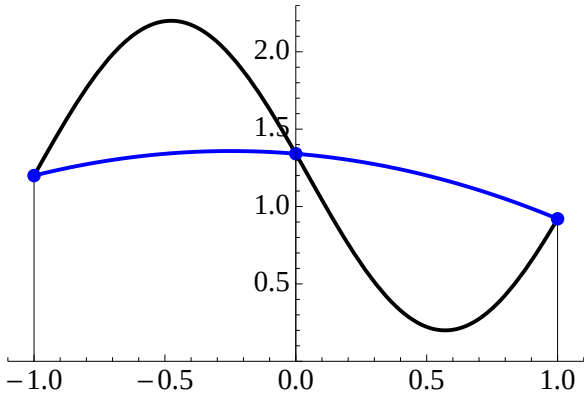
Reconstruct
(Interpolate)
$\mathcal{I}$



Reconstruction of physical quantities

Nodal quantities: p=2





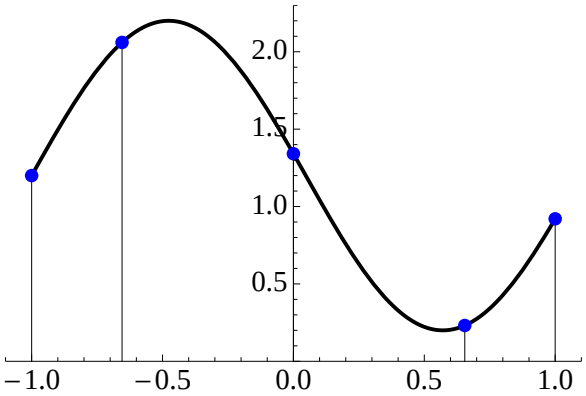
Reduce
(Discretize)
$\mathcal{R}$

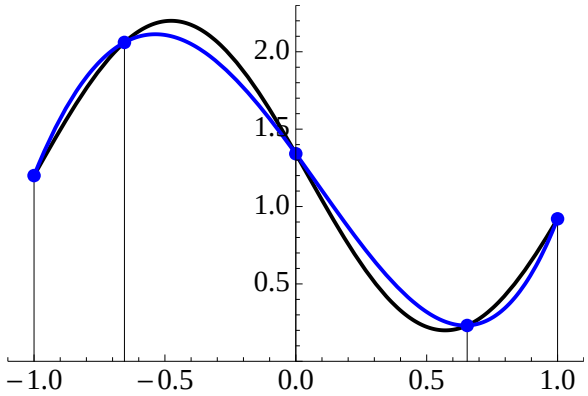
Reconstruct
(Interpolate)
$\mathcal{I}$

$$[\widetilde{\varphi}_i] = [\varphi(-1), \varphi(0), \varphi(1)]$$

Reconstruction of physical quantities

Nodal quantities: p=4





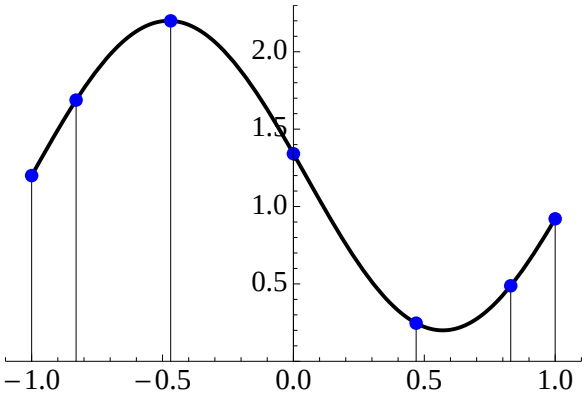
Reduce
(Discretize)
$\mathcal{R}$

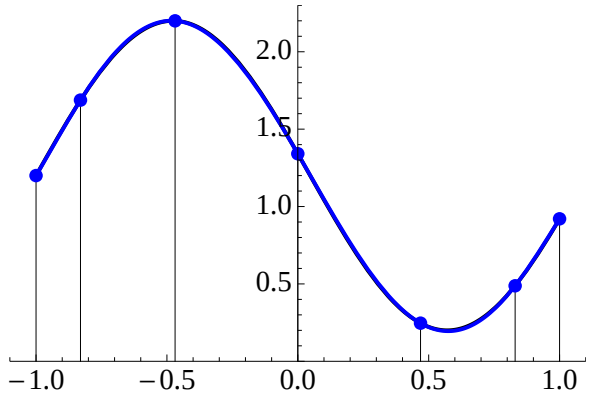
$$[\widetilde{\varphi}_i] = [\varphi(-1), (\cdots), \varphi(1)]$$

Reconstruct
(Interpolate)
$\mathcal{I}$

Reconstruction of physical quantities

Nodal quantities: p=6





Reduce
(Discretize)
$\mathcal{R}$

Reconstruct
(Interpolate)
$\mathcal{I}$

$$[\widetilde{\varphi}_i] = [\varphi(-1), (\cdots), \varphi(1)]$$

Reconstruction of physical quantities

Volume quantities: edge functions

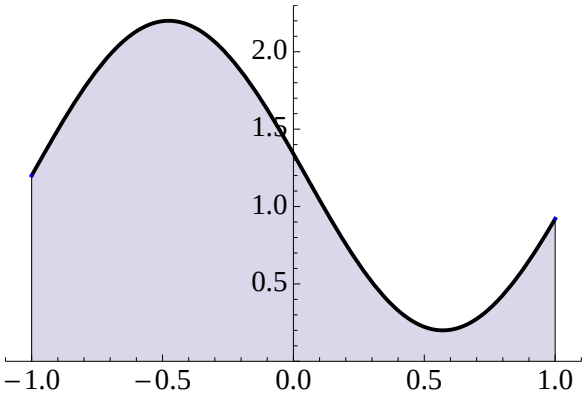
$$\int_{x_j}^{x_{j+1}} e_i^p(x) = \delta_{ij}$$

$$e_i^p(x) = -\sum_{n=1}^{i} \frac{d}{dx} h_n^{p+1}(x)$$

$i, j = 1, \ldots, p$

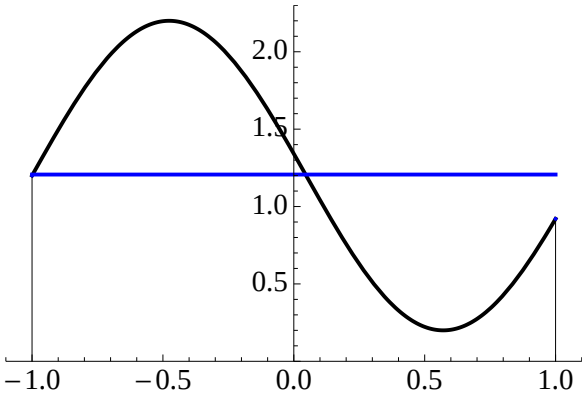
Reconstruction of physical quantities

Volume quantities: p=0



Reduce
(Discretize)
$\mathcal{R}$

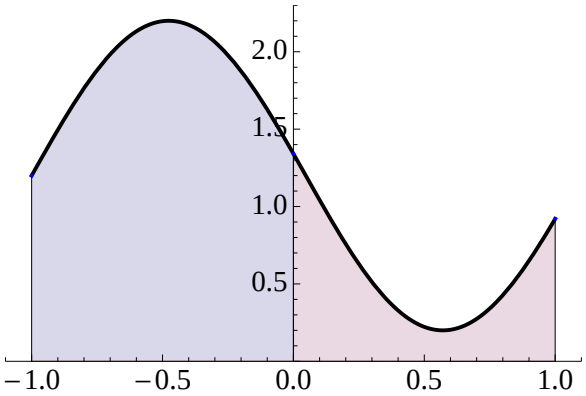
$$[\widetilde{\varphi}_i] = [\overline{\varphi}_1, \overline{\varphi}_2]$$

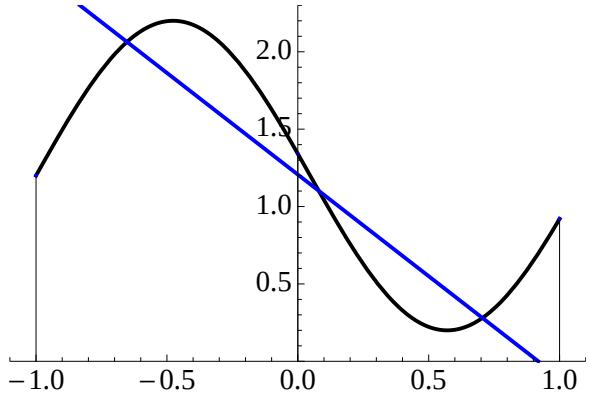


Reconstruct
(Histopolate)
$\mathcal{I}$

Reconstruction of physical quantities

Volume quantities: p=1





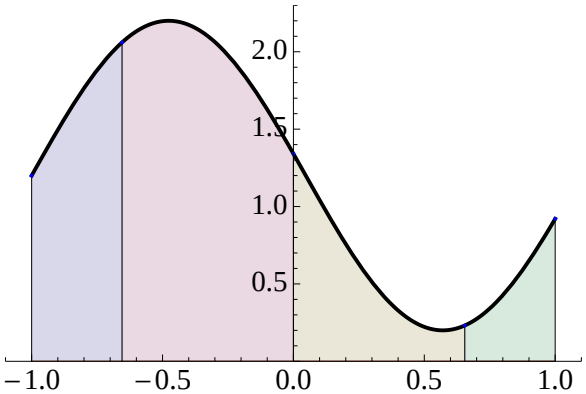
Reduce
(Discretize)
$\mathcal{R}$

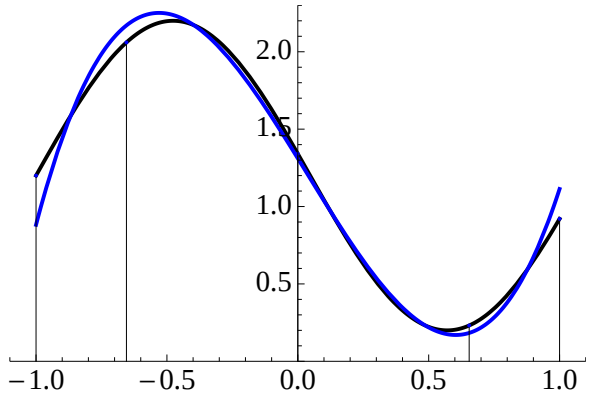
$$[\widetilde{\varphi}_i] = [\overline{\varphi}_1, \overline{\varphi}_2, \overline{\varphi}_3]$$

Reconstruct
(Histopolate)
$\mathcal{I}$

Reconstruction of physical quantities

Volume quantities: p=3





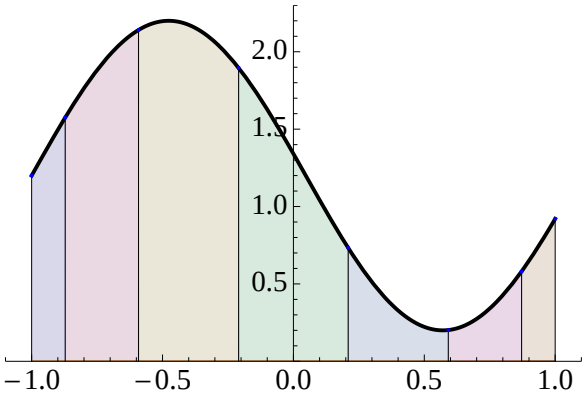
Reduce
(Discretize)
$\mathcal{R}$

Reconstruct
(Histopolate)
$\mathcal{I}$

$$[\widetilde{\varphi}_i] = [\overline{\varphi}_1, (\cdots), \overline{\varphi}_4]$$

Reconstruction of physical quantities

Volume quantities: p=6



Reduce
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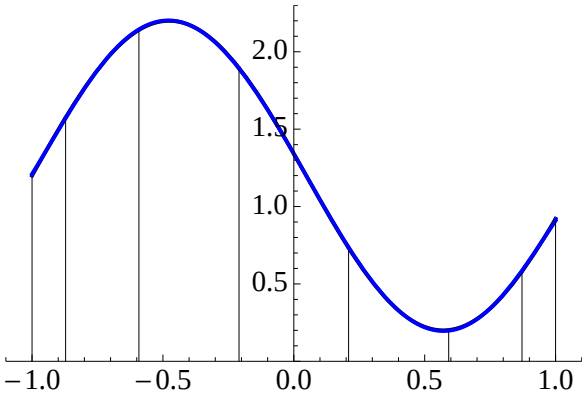
$$[\widetilde{\varphi}_i] = [\overline{\varphi}_1, (\cdots), \overline{\varphi}_7]$$



Reconstruct
(Histopolate)
$\mathcal{I}$

Higher dimensions: (2D case)

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Nodal quantities:

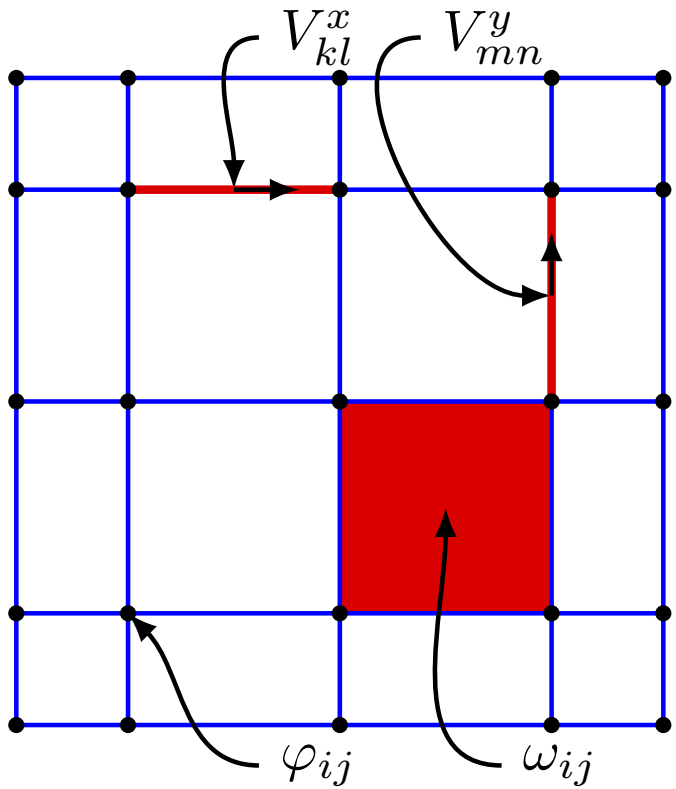
$$\varphi(x,y) \approx \widetilde{\varphi}^p(x,y) = \sum_{ij} \varphi_{ij} h_i^p(x) h_j^p(y)$$

Vector quantities:

$$\vec{V}(x,y) \approx \widetilde{\vec{V}}^p(x,y) = \sum_{ij} \overline{V}_{ij}^x e_i^{p-1}(x) h_j^p(y) \vec{e}_x + \sum_{ij} \overline{V}_{ij}^y h_i^p(x) e_j^{p-1}(y) \vec{e}_y$$

Volume quantities:

$$\omega(x,y) \approx \widetilde{\omega}^p(x,y) = \sum_{ij} \overline{\omega}_{ij} e_i^{p-1}(x) e_j^{p-1}(y)$$



Problem: how to transform from nodal to volume dof's?

Find $\widetilde{\star} : [\widetilde{\varphi}_i^0] \mapsto [\widetilde{\omega}_i^n] = \widetilde{\star}[\widetilde{\varphi}_i^0]$

Example: 1D case (n=1)

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Example: 1D case (n=1)

$$\widetilde{\varphi}^0 = \sum_i \varphi_i h_i(x)$$

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$$\widetilde{\varphi}^0 = \sum_i \varphi_i h_i(x)$$

$$\widetilde{\star}\widetilde{\varphi}^0 = \sum_i \varphi_i \star h_i(x)$$

$$\widetilde{\star}\widetilde{\varphi}^0 = \sum_i \sum_k H_{ik}\varphi_k e_i(x)$$

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$$\langle \sum_i \varphi_i \star h_i(x) - \sum_i \sum_k H_{ik}\varphi_k e_i(x), e_j \rangle = 0$$

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Example: 1D case (n=1)

$$\langle \sum_i \varphi_i \star h_i(x) - \sum_i \sum_k H_{ik} \varphi_k e_i(x), e_j \rangle = 0$$

$$\sum_i \varphi_i \langle \star h_i(x), e_j \rangle - \sum_i \sum_k H_{ik} \varphi_k \langle e_i(x), e_j \rangle = 0$$

Problem: how to transform from nodal to volume dof's?

Find $\widetilde{\star} : [\widetilde{\varphi}_i^0] \mapsto [\widetilde{\omega}_i^n] = \widetilde{\star}[\widetilde{\varphi}_i^0]$

Example: 1D case (n=1)

$$\langle \sum_i \varphi_i \star h_i(x) - \sum_i \sum_k H_{ik} \varphi_k e_i(x), e_j \rangle = 0$$

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Vector calculus vs Differential geometry

Vector calculus vs Differential geometry

$\nabla, \nabla\times, \nabla\cdot$ $\qquad$ d

Vector calculus vs Differential geometry

$$\nabla, \nabla\times, \nabla\cdot$$

$$\left.\begin{aligned}
\int_{\Omega^1} \nabla\varphi \cdot \mathrm{d}\vec{s} &= \varphi(\Omega^1_B) - \varphi(\Omega^1_A) \\
\int_{\Omega^2} (\nabla\times\vec{V}) \cdot \vec{n}\,\mathrm{d}S &= \int_{\partial\Omega^2} \vec{V}\cdot\mathrm{d}\vec{s} \\
\int_{\Omega^3} (\nabla\cdot\vec{V})\mathrm{d}V &= \int_{\partial\Omega^3} \vec{V}\cdot\vec{n}\,\mathrm{d}S
\end{aligned}\right\}$$

$$\mathrm{d}$$

$$\int_{\Omega^{k+1}} \mathrm{d}\alpha^k = \int_{\partial\Omega^{k+1}} \alpha^k$$

Vector calculus vs Differential geometry

$$\nabla, \nabla\times, \nabla\cdot \qquad\qquad \mathrm{d}$$

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$$\int_{\Omega^{k+1}} \mathrm{d}\alpha^k = \int_{\partial\Omega^{k+1}} \alpha^k$$

$$\left.\begin{aligned} \varphi^0 &\overset{\star}{\longleftrightarrow} \omega^n \\ \vec{V}_L &\overset{\star}{\longleftrightarrow} \vec{V}_F \end{aligned}\right\}$$

$$\begin{array}{ccccccccccccc} \mathbb{R} & \longrightarrow & \Lambda^0 & \xrightarrow{\mathrm{d}} & \Lambda^1 & \xrightarrow{\mathrm{d}} & \Lambda^2 & \xrightarrow{\mathrm{d}} & \Lambda^3 & \longrightarrow & 0 \\ & & \downarrow\star & & \downarrow\star & & \downarrow\star & & \downarrow\star & & \\ 0 & \longleftarrow & \tilde{\Lambda}^3 & \xleftarrow[\mathrm{d}]{} & \tilde{\Lambda}^2 & \xleftarrow[\mathrm{d}]{} & \tilde{\Lambda}^1 & \xleftarrow[\mathrm{d}]{} & \tilde{\Lambda}^0 & \longleftarrow & \mathbb{R} \end{array}$$

Vector calculus vs Differential geometry

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$$\vec{J} = q\vec{V} \qquad\qquad \iota_{\vec{V}} \star q^0 = \star(\varphi^0 \wedge \vec{V}^\flat)$$

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Discrete advection equation

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$$\left\{ \begin{array}{rl} q\vec{V} & = \vec{J} \\ \overline{\nabla} \cdot \vec{J} & = 0 \end{array} \right. \qquad \left. \begin{array}{rl} \iota_{\vec{V}} \star q^0 & = j^{n-1} \\ \mathrm{d} j^{n-1} & = 0 \end{array} \right\} = \mathcal{L}_{\vec{V}} \star q^0$$

Discrete advection equation

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$$\left\{ \begin{array}{rl} [IV_{ij}][q_j] & = [j_i] \\ [D_{ij}]\,[j_i] & = 0 \end{array} \right.$$

$$[IV_{ij}] = [\langle \iota_{\vec{V}} \star h_k(x,y), e_i(x,y)\rangle_{ki}]^{-t}[\langle e_j, e_k\rangle_{jk}]^t = [\overline{E}_{ij}]^{-t}[E_{ij}]^t$$

Discrete advection equation

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$$\left\{ \begin{array}{rl} [E_{ij}]^t[q_j] - [\overline{E}_{ij}]^t[j_i] & = 0 \\ [D_{ij}]\,[j_i] & = 0 \end{array} \right.$$

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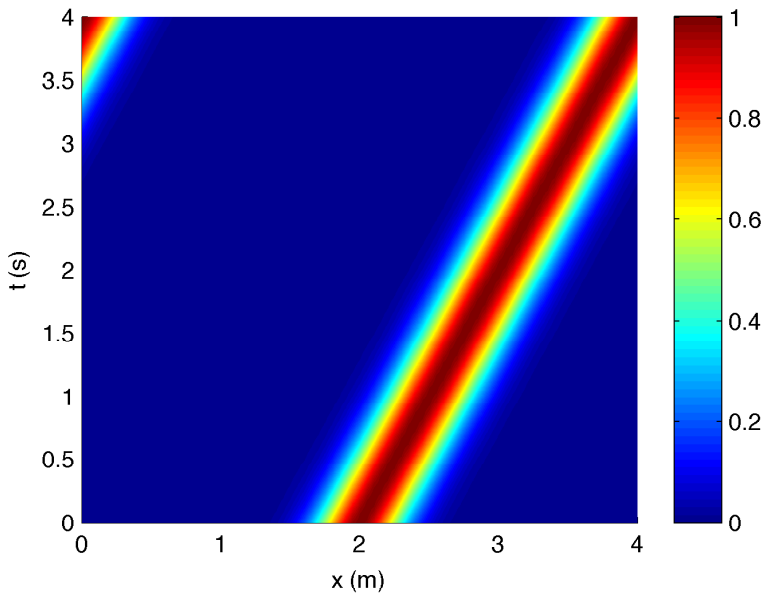
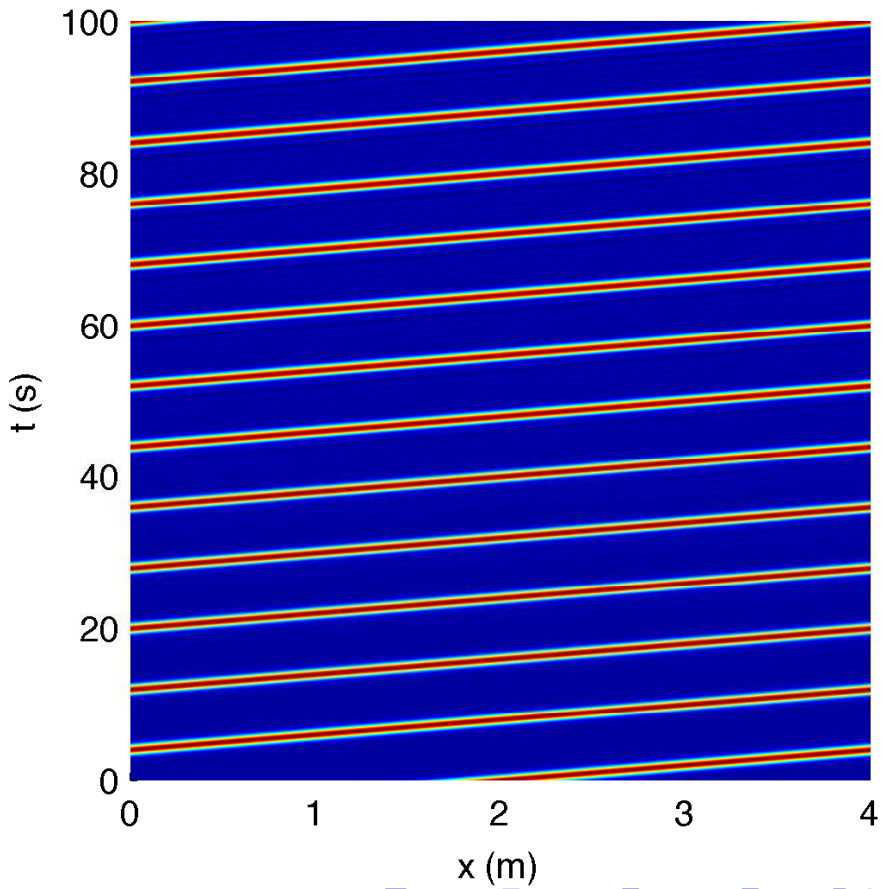
Results

Results

Cosine hill: p=4, v=0.5, $\Delta x = 0.4$, $\Delta t = 0.4$





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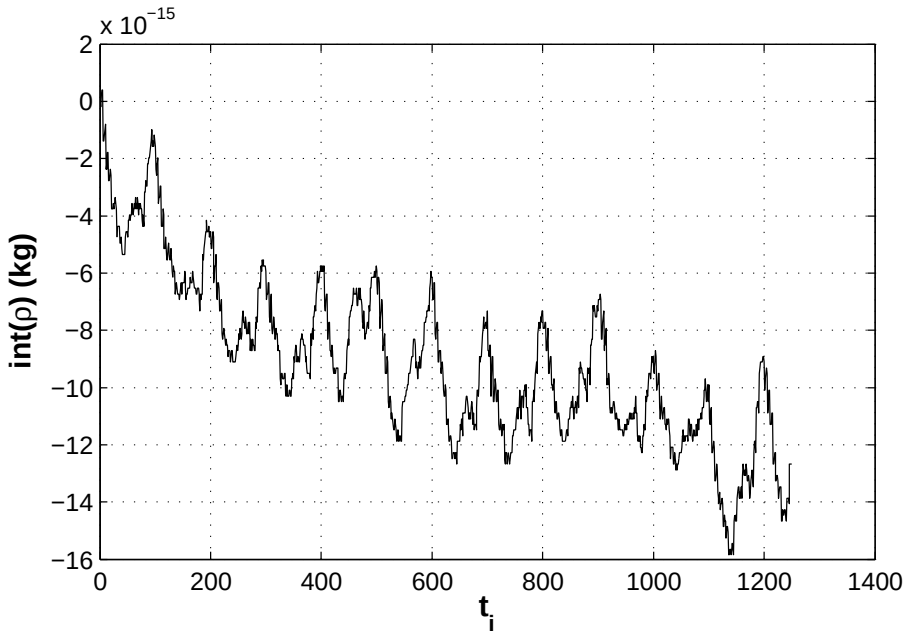
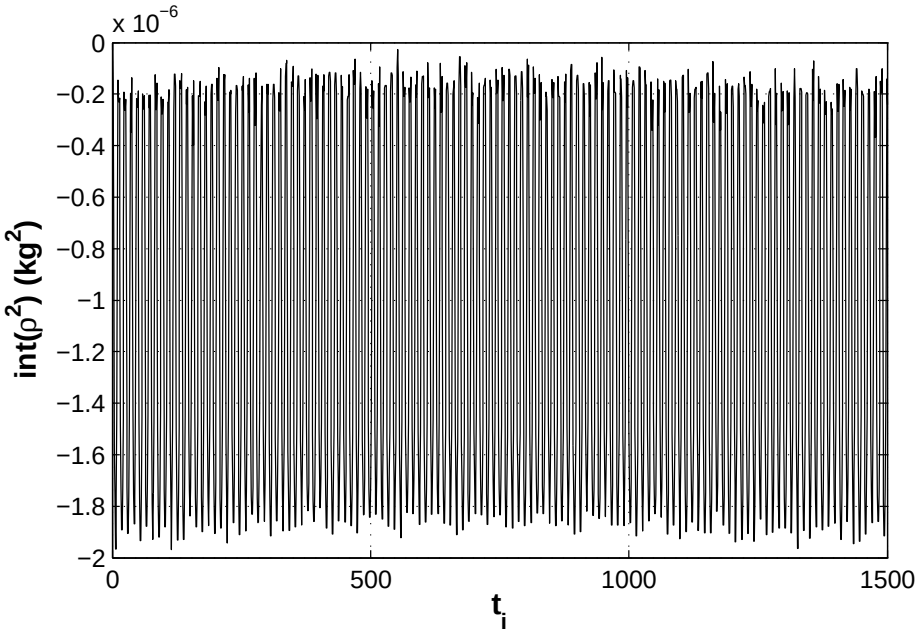
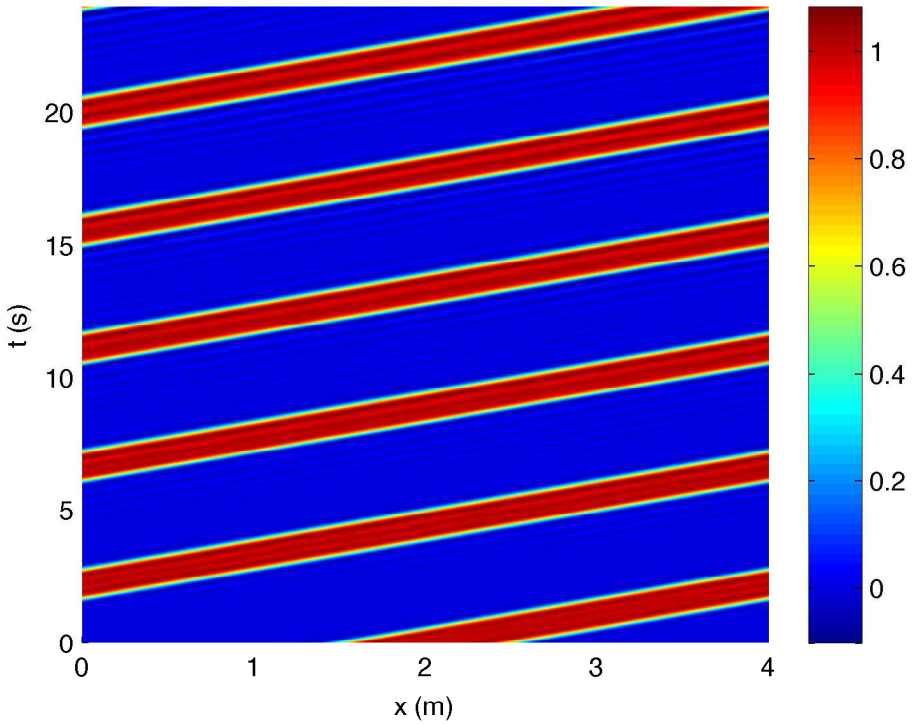




Results

Square wave: p=10, v=0.9, $\Delta x = 1.0$, $\Delta t = 1.0$



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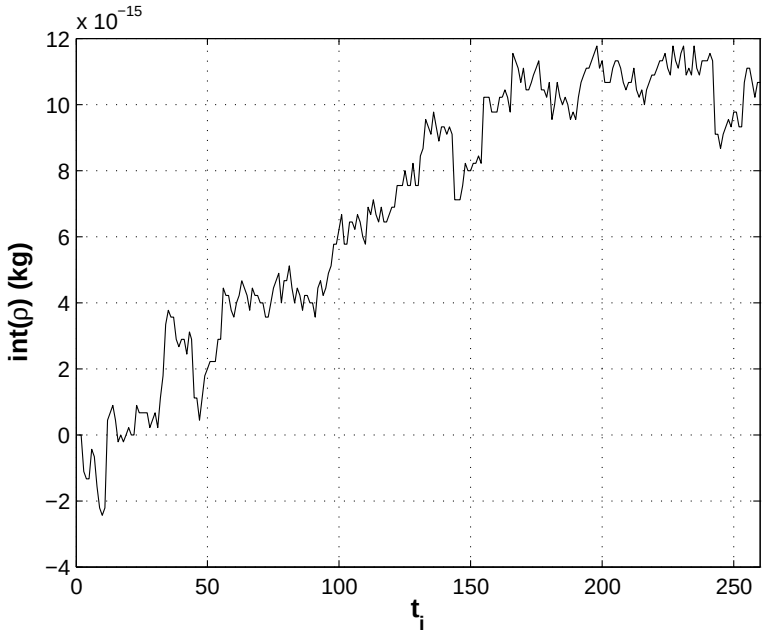
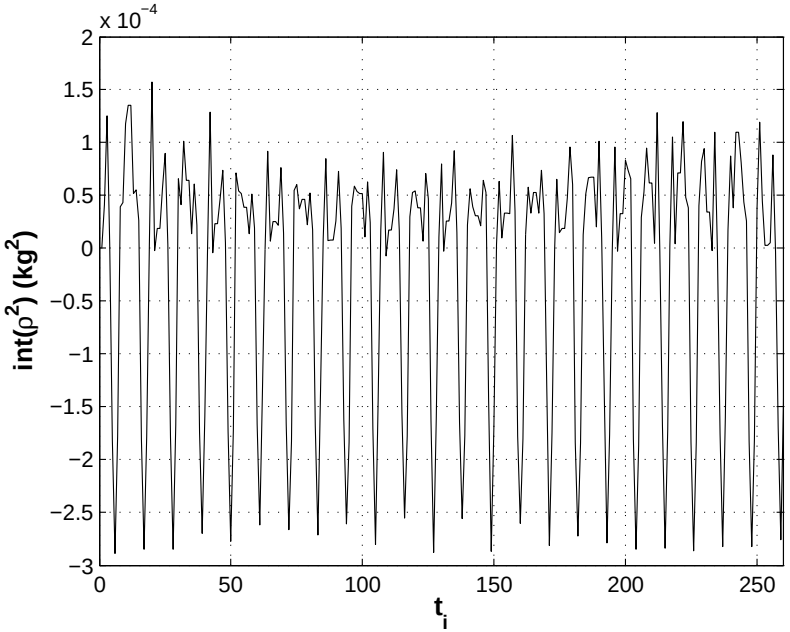
Square wave: p=10, v=0.9, $\Delta x = 1.0$, $\Delta t = 1.0$





Conclusions and future work

- Discretize advected quantity as 1-form (space-density)
- Integrate with diffusion
- Deformed domains
- Non-linear advection equation (Burger's equation)

End

Acknowledgements

The authors would like to acknowledge Foundation for Science and Technology (Portugal) for the funding given by the PhD grant SFRH/BD/36093/2007.