Spectral Mimetic Methods on Quadrilaterals

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Compatible and Innovative Discretizations for PDE's, 2009

Outline of presentation

- Motivation: exact representation of physical laws;
- Concepts from differential geometry;
- Concepts from algebraic topology;
- Construction of Hodge operators;
- Example: Application to the Poisson equation.

Differential forms[1]: representation of physical quantities

Let $\mathcal{M} \subset \mathbb{R}^n$ a differentiable manifold, then
k-form $\omega^k \in \Lambda^k$: a rank-k, anti-symmetric, tensor field over $\mathcal{M}$ ▸ Example diff. form

$$\omega^k \ : \ \underbrace{T_xM \times \cdots \times T_xM}_{k \text{ copies}} \longrightarrow \mathbb{R} \, ,$$

$$\omega^k(\dots, v_i, \dots, v_j, \dots) = -\omega^k(\dots, v_j, \dots, v_i, \dots)$$

Wedge product: Let $\omega^k \in \Lambda^k$ and $\omega^l \in \Lambda^l$ then

$$\wedge \ : \ \Lambda^k \times \Lambda^l \ \longrightarrow : \Lambda^{k+l}$$

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[1] Cartan [?], Spivak [?], Flanders [?]

Differential forms: intrinsic connection with geometry

Under integration, one can state a duality pairing between k-forms and k-manifolds:

$$\int_{\Omega_k} \omega^k = \langle \omega^k, \Omega_k \rangle \in \mathbb{R}$$

Leads to an instrisic connection between differential forms and geometrical objects (in $\mathbb{R}^3$):

- 0-forms ⟶ Points
- 1-forms ⟶ Lines
- 2-forms ⟶ Surfaces
- 3-forms ⟶ Volumes

Operators: the exterior derivative d

The exterior derivative d, in a n-dimensional space, is a mapping:

$$\mathrm{d} : \Lambda^k \mapsto \Lambda^{k+1}, \qquad k = 0, 1, \ldots, n-1,$$

which satisfies:

$$\mathrm{d}\left(\omega^k \wedge \alpha^l\right) = \mathrm{d}\omega^k \wedge \alpha^l + (-1)^k \omega^k \wedge \mathrm{d}\alpha^l, \qquad k + l < n$$

and:

$$\mathrm{d}\mathrm{d}\omega^k = 0, \qquad \forall \omega^k \in \Lambda^k, \quad k < n-1$$

Leads to the exact sequence (de Rham complex):

$$\mathbb{R} \hookrightarrow \Lambda^0(\Omega) \overset{d}{\mapsto} \Lambda^1(\Omega) \overset{d}{\mapsto} \Lambda^2(\Omega) \overset{d}{\mapsto} \Lambda^3(\Omega) \overset{d}{\mapsto} 0$$

In $\mathbb{R}^3$: $d^0 \leftrightarrow \nabla$, $d^1 \leftrightarrow \nabla\times$ and $d^2 \leftrightarrow \nabla\cdot$.

Operators: exterior derivative d

Stokes Theorem: Let Ω_{k+1} be a $k+1$-dimensional manifold and $\omega \in \Lambda^k$ then

$$\int_{\partial\Omega_{k+1}} \omega^k = \int_{\Omega_{k+1}} d\omega^k$$

▸ Example generalized Stokes

$$\begin{tikzcd}
\int_{\partial\Omega_{k+1}} \omega^k \arrow[d, "\text{duality pairing}"'] & = & \int_{\Omega_{k+1}} d\omega^k \arrow[d, "\text{duality pairing}"] \\
\langle \omega^k, \partial\Omega_{k+1} \rangle & = & \langle \mathrm{d}\omega^k, \Omega_{k+1} \rangle
\end{tikzcd}$$

d by duality pairing is the formal adjoint of ∂.

Operators: the Hodge-$\star$ operator

What about ∇^2?

$$\nabla^2 = \nabla \cdot \nabla \neq \mathrm{d} \circ \mathrm{d} = 0$$

What about $\mathrm{d}^{(0)} \circ \mathrm{d}^{(2)}$? No because:

$$\left.\begin{array}{l} \mathrm{d}^{(0)} : \Lambda^0 \longrightarrow \Lambda^1 \\ \mathrm{d}^{(n-1)} : \Lambda^{n-1} \longrightarrow \Lambda^n \end{array}\right\} \Longrightarrow \mathcal{R}(d^{(0)}) \not\subset \mathcal{D}(d^{(n-1)})$$

An additional operator $\star$ is needed, such that:

$$\star : \Lambda^k \mapsto \Lambda^{n-k}$$

Then:

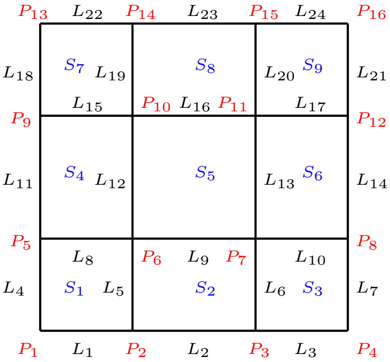
$$\nabla^2 \leftrightarrow \mathrm{d}^{(n-1)} \star \mathrm{d}^{(0)}$$

And enables the construction of the de Rham complex, for example in $\mathbb{R}^3$:

$$\begin{tikzcd}
\mathbb{R} \arrow[r] & \Lambda^0 \arrow[r, "\mathrm{d}"] \arrow[d, "\star"'] & \Lambda^1 \arrow[r, "\mathrm{d}"] \arrow[d, "\star"'] & \Lambda^2 \arrow[r, "\mathrm{d}"] \arrow[d, "\star"'] & \Lambda^3 \arrow[r] \arrow[d, "\star"'] & 0 \\
0 & \tilde{\Lambda}^3 \arrow[l] & \tilde{\Lambda}^2 \arrow[l, "\mathrm{d}"] & \tilde{\Lambda}^1 \arrow[l, "\mathrm{d}"] & \tilde{\Lambda}^0 \arrow[l, "\mathrm{d}"] & \mathbb{R} \arrow[l]
\end{tikzcd}$$

k-chains: the discrete k-manifolds

A p-chain is a formal sum of oriented p-cells. 0-cells are points, 1-cells are curves, 2-cells surfaces and 3-cells represent volumes. The collection of all p-chains is called a cell complex.



Cell complex K and labeling of the 0-cells, 1-cells and 2-cells

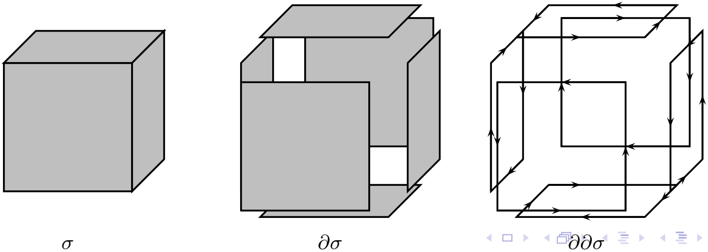
k-chains: the discrete k-manifolds

A p-chain is a formal sum of oriented p-cells.

$$C^{(p)} = \sum_{i=1}^{N_{p-cell}} m_i \sigma_i^{(p)}$$

If σ is a p-cell, then its boundary constitutes a $(p-1)$-chain denoted by $\partial\sigma$
The boundary of the boundary is empty for every p-cell

$$\boxed{\partial\partial\sigma \equiv 0}$$



k-cochains: the discrete k-forms

With every k-cell we can associate a value from a field $\mathbb{F}$. Assume $\mathbb{F} = \mathbb{R}$,

$$\sigma^k \; : \; \sigma_k \; \longrightarrow \; \mathbb{R}$$

By doing so, we can associate with every k-chain a so-called k-co-chain

$$C^k = \sum_{i=1}^{N_{k-cell}} \alpha_i \sigma^{i,(k)}$$

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Mimetizing the duality pairing between k-forms and k-manifolds we obtain the duality pairing between co-chains and chains that also satisfies all the properties of geometric integration

$$\langle \omega^k, \Omega_k \rangle \in \mathbb{R} \longrightarrow \left\langle C^k, C_k \right\rangle \in \mathbb{R}$$

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$$\langle \omega^k, \Omega_k \rangle \in \mathbb{R} \longrightarrow \left\langle C^k, C_k \right\rangle \in \mathbb{R}$$

Note that both chains and co-chains are metric free.

Co-boundary operator, δ: the discrete d

Let C_{p+1} be a $(p+1)$-chain, then its boundary ∂C_{p+1} is a p-chain, then formally we can write

$$\langle C^p, \partial C_{p+1} \rangle = \langle \delta C^p, C_{p+1} \rangle$$

where δ is the formal adjoint of the boundary operator ∂.

This adjoint relation is the discrete analogue of the generalized Stokes theorem. Remember that the Stokes Theorem was given by

$$\left\langle \omega^k, \partial \Omega_{k+1} \right\rangle = \left\langle d\omega^k, \Omega_{k+1} \right\rangle$$

δ is a discrete version of the exterior derivative and is called the coboundary operator

$$\delta \; : \; C^p \; \longrightarrow \; C^{p+1}$$

Note that $\delta\delta C^p \equiv 0$ just like $dd\omega^k \equiv 0$, since

$$\langle \delta\delta C^p, C_{p+2} \rangle = \langle \delta C^p, \partial C_{p+2} \rangle = \langle C^p, \partial\partial C_{p+2} \rangle \equiv 0$$

▸ The action of δ ▸ The incidence matrix

Continuous – Discrete parallel

Continuous	Discrete
k-manifolds	k-chains
k-forms	k-co-chains
d	δ
∂	∂

Continuous – Discrete parallel

Continuous	Discrete
k-manifolds	k-chains
k-forms	k-co-chains
d	δ
∂	∂

Missing link!

Continuous – Discrete parallel

Continuous	Discrete
k-manifolds	k-chains
k-forms	k-co-chains
d	δ
∂	∂
$\star$	??

Different ways to define the discrete Hodge-$\star$ operator.

The 2D Poisson equation: the case study

The equation

$$\left\{\begin{array}{rcl} \nabla\phi & = & \mathbf{v} \\ \nabla\cdot\mathbf{q} & = & f \\ \mathbf{q} & = & \rho\mathbf{u} \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} \mathrm{d}\phi^0 & = & u^1 \\ \mathrm{d}\tilde{q}^1 & = & \tilde{f}^2 \\ \tilde{q}^1 & = & \star_\rho u^1 \end{array}\right.$$

The 2D Poisson equation: the case study

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The Tonti diagram

$$\begin{tikzcd}
\phi^0 \arrow[d, "\mathrm{d}"'] & \tilde{f}^2 \\
u^1 \arrow[r, "\star_\rho"'] & \tilde{q}^1 \arrow[u, "\mathrm{d}"']
\end{tikzcd}$$

The discrete Hodge-$\star$: ways to build it

- Dual grid methods:
    - Support operator method
    - Galerkin projection method
- Primal-dual grid method:
    - Weak material laws

The discrete Hodge-$\star$: reduction operator $\mathcal{R}$

The Reduction Operator $\mathcal{R}$ is naturally given by

$$\boxed{\mathcal{R} \, : \, \Lambda^p \, \longrightarrow \, C^p \ , \ \sigma^p = \mathcal{R}(\omega^p) := \int_{\sigma_p} \omega^p}$$

Note that this operation requires that the p-cell σ_p is no longer considered as topological element, but now acquires metrical properties.

It is quite straightforward to show that

$$\boxed{\mathcal{R} d = \delta \mathcal{R}}$$

so external derivative and coboundary operator commute with $\mathcal{R}$. In principle, no error is introduced by the application of $\mathcal{R}$.

The discrete Hodge-$\star$: interpolation operator $\mathcal{I}$

The Interpolation Operator $\mathcal{I}$ must satisfy two basic conditions

$$\boxed{\mathcal{R}\mathcal{I} = Id}$$

i.e. $\mathcal{I}$ is a right-inverse of $\mathcal{R}$ and

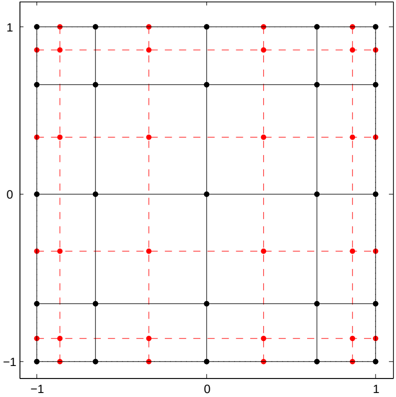
$$\boxed{\mathcal{I}\mathcal{R} = Id + O(h^s)}$$

i.e. $\mathcal{I}$ is an approximate left-inverse of $\mathcal{R}$. Furthermore, we require that

$$\boxed{d\mathcal{I} = \mathcal{I}\delta}$$

Support operator method: the degrees of freedom

Consider the dual spectral element mesh consisting of a Gauss-Lobatto-Legendre (GLL) points and Extended Gauss-Legendre (EGL) points.



Dual cell complex: GLL-grid (black) and EGL-grid (red)

Support operator method: the degrees of freedom

GLL - nodal interpolation

$$h_i(\xi) = \frac{\left(1-\xi^2\right) L_N'(\xi)}{N(N+1)L_N(\xi_i)(\xi_i - \xi)} \ , \quad i = 0, \ldots, N$$

$L_N(\xi)$ Legendre polynomial of degree N, $-1 = \xi_0 < \xi_1 < \cdots < \xi_{N-1} < \xi_N = 1$, zeros of $L_N'(\xi)$.

GLL - edge interpolation

$$e_i(\xi) = -\sum_{k=0}^{i-1} dh_k(\xi)$$

$$\int_{\xi_{k-1}}^{\xi_k} e_i(\xi) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

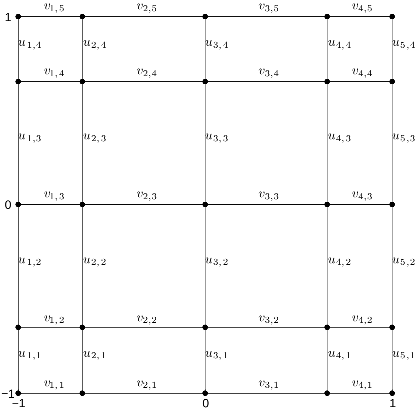
Histopolation

EGL - nodal interpolation

$$\tilde{h}_i(\xi) = \begin{cases} \frac{(-1)^N}{2}\,(1-\xi)\,L_N(\xi) & \text{if } i = 0 \\ \frac{\left(1-\xi^2\right) L_N(\xi)}{\left(1-\tilde{\xi}_i^2\right) L_N'(\xi_i)(\xi - \tilde{\xi}_i)} & \text{if } 1 \leq i \leq N \\ \frac{1}{2}\,(1+\xi)\,L_N(\xi) & \text{if } i = N+1 \end{cases}$$

Support operator method: the degrees of freedom

Suppose the normal fluxes $\tilde{u}^1$ are given in the grid as



Location of the normal velocity components on GLL grid

Support operator method: the degrees of freedom

With these basis functions we have that the interpolation normal velocity components are given by

$$\mathcal{I}\mathcal{R}\star\mathbf{u}(\xi,\eta) = -\sum_{i=1}^{N}\sum_{j=0}^{N}\star v_{i,j}e_i(\xi)h_j(\eta) + \sum_{i=0}^{N}\sum_{j=1}^{N}\star u_{i,j}h_i(\xi)e_j(\eta)$$

Application of the exterior derivative gives

$$\begin{aligned} d\mathcal{I}\mathcal{R}\star\mathbf{u}(\xi,\eta) &= -\sum_{i=1}^{N}\sum_{j=0}^{N}\star v_{i,j}e_i(\xi)dh_j(\eta) + \sum_{i=0}^{N}\sum_{j=1}^{N}\star u_{i,j}dh_i(\xi)e_j(\eta) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\star u_{i,j} + \star v_{i,j} - \star u_{i-1,j} - \star v_{i,j-1}\right)e_i(\xi)e_j(\eta) \\ &= \mathcal{I}\delta\mathcal{R}\star\mathbf{u}\;, \end{aligned}$$

So we have the commuting property between exterior derivative and the coboundary operator

$$\boxed{d\mathcal{I} = \mathcal{I}\delta}$$

Support operator method: how to get the Hodge-$\star$

Now that we have discretized the divergence operator, we can implicitly define the gradient operator using

$$\begin{aligned}\int_\Omega \left(\star d\phi^{(0)}, \star u^{(1)}\right) \omega_n &= \int_\Omega d\phi^{(0)} \wedge \star u^{(1)} \\ &= \int_{\partial\Omega} \phi^{(0)} \wedge \star u^{(1)} - \int_\Omega \phi^{(0)} \wedge d \star u^{(1)} \,,\end{aligned}$$

as in the support operator method for finite difference methods, see for instance Hyman, J., Shaskov, M., Steinberg, S.: *The numerical solution of diffusion problems in strongly heterogeneous non-isotropic materials*. Journal of Computational Physics **132**, 130 - 148, 1997

The condition number of the resulting discrete Hodge operator satisfies

$$1 \leq \text{cond}\left(H^d\right) \leq 2$$

See also Nicolas Robidoux, *Polynomial Histopolation, superconvergent degrees of freedom and pseudospectral discrete Hodge operators*, to appear.

Support operator method: the discrete formulation

$$dq^{(1)} = f^{(2)}$$

Support operator method: the discrete formulation

$$dq^{(1)} = f^{(2)} \qquad \Longleftrightarrow \qquad \mathbb{E}^{(2,1)}q^1 = f^2$$

Support operator method: the discrete formulation

$$\begin{aligned} dq^{(1)} &= f^{(2)} \\ u^{(1)} &= d\phi^{(0)} \end{aligned} \qquad \Longleftrightarrow \qquad \mathbb{E}^{(2,1)} q^1 = f^2$$

Support operator method: the discrete formulation

$$\begin{aligned} dq^{(1)} &= f^{(2)} & \Longleftrightarrow && \mathbb{E}^{(2,1)}q^1 &= f^2 \\ u^{(1)} &= d\phi^{(0)} & \Longleftrightarrow && u^1 &= \left(\mathbb{E}^{(2,1)}\right)^T \phi^0 \end{aligned}$$

Support operator method: the discrete formulation

$$\begin{aligned} dq^{(1)} &= f^{(2)} & \Longleftrightarrow \quad & \mathbb{E}^{(2,1)} q^1 = f^2 \\ u^{(1)} &= d\phi^{(0)} & \Longleftrightarrow \quad & u^1 = \left(\mathbb{E}^{(2,1)}\right)^T \phi^0 \\ q^{(1)} &= \star u^{(1)} & & \end{aligned}$$

Support operator method: the discrete formulation

$$\begin{aligned} dq^{(1)} &= f^{(2)} && \Longleftrightarrow && \mathbb{E}^{(2,1)}q^1 = f^2 \\ u^{(1)} &= d\phi^{(0)} && \Longleftrightarrow && u^1 = \left(\mathbb{E}^{(2,1)}\right)^T \phi^0 \\ q^{(1)} &= \star u^{(1)} && \Longleftrightarrow && q^1 = \mathbf{H}^d u^1 \end{aligned}$$

Support operator method: the discrete formulation

$$\begin{aligned} dq^{(1)} &= f^{(2)} & \Longleftrightarrow && \mathbb{E}^{(2,1)}q^1 &= f^2 \\ u^{(1)} &= d\phi^{(0)} & \Longleftrightarrow && u^1 &= \left(\mathbb{E}^{(2,1)}\right)^T \phi^0 \\ q^{(1)} &= \star u^{(1)} & \Longleftrightarrow && q^1 &= \mathbf{H}^d u^1 \end{aligned}$$

The first two equations are purely topological and will be the same on topologically equivalent meshes. These two equations coincide with finite volume discretizations on a staggered grid.
The last equation – the constitutive equation – couples the solution on the two grids via the discrete Hodge operator $\mathbf{H}^d$. All the metric properties (angles, distances, areas, etc) are contained in $\mathbf{H}^d$.

$$\boxed{\mathbb{E}^{(2,1)}\mathbf{H}^d\left(\mathbb{E}^{(2,1)}\right)^T \phi^0 = f^2}$$

Support operator method: the equivalence with LS

Lemma

The same discretization is obtained in terms of a least-squares formulation. If we minimize the difference between $\star\mathcal{G}\phi$ and $\star d\phi$ as

Find $\star\,\mathcal{G}\phi \in \Lambda^{n-1}\left(\Omega; L^2\right)$ *such that for* $\phi \in \Lambda^0\left(\Omega; L^2\right)$

$$*\mathcal{G}\phi = \arg \min_{v \in \Lambda^{n-1}\left(\Omega; L^2\right)} \frac{1}{2} \left\| v - \star d\phi \right\|^2_{L^2(\Omega)} \qquad (1)$$

Proof.

A necessary condition for a minimizer is

$$\begin{aligned} (v, w) &= (*d\phi, w) = \int_\Omega d\phi \wedge w \\ &= \int_{\partial\Omega} \phi \wedge w - \int_\Omega \phi \wedge dw\ , \quad \forall w \in \Lambda^{n-1}\left(\Omega; L^2\right) \end{aligned}$$

The trivial norm-equivalence

$$C_1 \|v\|_{L^2(\Omega)} \le \|v\|_{L^2(\Omega)} \le C_2 \|v\|_{L^2(\Omega)}\ , \quad \text{for } C_1 = C_2 = 1\ ,$$

Galerkin projection method

Primal-dual grid method: the role of least-squares

The idea

- Impose the constitutive equation weakly
- Hodge-$\star$ operator defined implicitly
- Minimize local discrepancy between dual variables

The implementation

Seek (ϕ_h^0, v_h^1, q_h^1) in $\Lambda_h^0 \times \Lambda_h^1 \times \Lambda_h^1$ such that

$$\mathcal{I}(\phi_h^0, v_h^1, q_h^1) = \frac{1}{2}\left(\| \star q_h^1 + v_h^1\|_0^2 + \|\mathrm{d}q_h^1 - f^2\|_0^2\right)$$

subject to: $\mathrm{d}\phi_h^0 = v_h^1$

Primal-dual grid method: the role of least-squares

If the subspaces Λ_h^0, Λ_h^1 and Λ_h^2 are chosen in such a way that they constitute a de Rham complex:

$$\mathbb{R} \longrightarrow \Lambda_h^0 \xmapsto{\mathrm{d}} \Lambda_h^1 \xmapsto{\mathrm{d}} \Lambda_h^2 \longmapsto 0$$

then $\mathrm{d}\phi_h^0 = v_h^1$ is satisfied exactly. The problem becomes:

$$\text{Seek } (\phi_h^0, q_h^1) \text{ in } \Lambda_h^0 \times \Lambda_h^1 \text{ such that}$$
$$\mathcal{I}(\phi_h^0, q_h^1) = \frac{1}{2}\left(\| \star q_h^1 + \mathrm{d}\phi^0\|_0^2 + \|\mathrm{d}q_h^1 - f^2\|_0^2\right)$$

In this way, the Hodge-$\star$ operator is implemented as L^2 projections between the different dual spaces.

Primal-dual grid method: the role of least-squares

Find adequate subspaces Λ_h^0, Λ_h^1 and Λ_h^2 must be specified. Since one will use a spectral/hp LS method, these spaces are defined as:

$$\begin{aligned}
\Lambda^0_{h,p} &= \operatorname{span}\left\{h_i^p(x)h_j^p(y)\right\} \quad , \quad i=0,\dots,p \quad j=0,\dots,p \\
\Lambda^1_{h,p} &= \operatorname{span}\left\{e_{i,j}^p(x,y)\right\} \quad , \quad i=1,\dots,p \quad j=0,\dots,p \\
\Lambda^2_{h,p} &= \operatorname{span}\left\{s_{i,j}^p(x,y)\right\} \quad , \quad i=1,\dots,p \quad j=1,\dots,p
\end{aligned}$$

- $h_{i,j}^p(x,y)$: i-th nodal cardinal basis functions over GLL points
- $e_{i,j}^p(x,y)$: i-th edge cardinal basis functions over GLL edges
- $s_{i,j}^p(x,y)$: i-th surface cardinal basis functions over GLL surfaces
- Degrees of freedom are located where they should be: at nodal points (for 0-forms), at edges (for 1-forms) and at surfaces (for 2-forms).
- Different continuity properties
- These subspaces constitute a de Rham complex

Example Differential forms

Without fancy mathematics we can say that a differential form is something we find underneath the integral sign, [?]. In 3D a 1 form looks something like

$$\omega^1(x,y,z) = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

A 2-form looks something like

$$\xi^2 = A(x,y,z)\,dxdy + B(x,y,z)\,dydz + C(x,y,z)\,dzdx$$

and a 3-form can be expressed as

$$\rho^3 = K(x,y,z)\,dxdydz$$

1-forms live on 1-dimensional geometric objects, 2-forms live on surfaces and 3-forms are associated with volumes. A 0-form is a scalar-valued function defined in points.

▸ Back

Example exterior derivative 0-form

Let ω be the 1-form given by

$$\omega^0(x,y,z) = f(x,y,z)$$

then the exterior derivative gives

$$d\omega^0(x,y,z) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

We can associate the vector $\mathrm{grad}\, f = (f_x, f_y, f_z)^T$ with the exterior derivative of $\omega^0 = f$. This is denoted by $\mathrm{grad}\, f =^\sharp d\omega^0$ or $d\omega^0 =^\flat \mathrm{grad}\, f$.

Example exterior derivative 1-form

Let ω be the 1-form given by

$$\omega^1(x,y,z) = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

then the exterior derivative gives

$$\begin{aligned} d\omega^1(x,y,z) \quad &= \quad \frac{\partial P}{\partial x} dx \wedge dx + \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial P}{\partial z} dz \wedge dx + \\ & \quad\quad \frac{\partial Q}{\partial x} dx \wedge dy + \frac{\partial Q}{\partial y} dy \wedge dy + \frac{\partial Q}{\partial z} dz \wedge dy + \\ & \quad\quad \frac{\partial R}{\partial x} dx \wedge dz + \frac{\partial R}{\partial y} dy \wedge dz + \frac{\partial R}{\partial z} dz \wedge dz \\ &= \quad \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy \end{aligned}$$

where we used $dx \wedge dy = -dy \wedge dx$, $dy \wedge dz = -dz \wedge dy$, etc. and $dx \wedge dx = dy \wedge dy = dz \wedge dz = 0$.

Example exterior derivative 2-form

Let ω be the 1-form given by

$$\omega^2(x,y,z) = A(x,y,z)\,dydz + B(x,y,z)\,dzdx + C(x,y,z)\,dxdy$$

then the exterior derivative gives

$$\begin{aligned}
d\omega^2(x,y,z) \quad &= \quad \frac{\partial A}{\partial x} dx \wedge dy \wedge dz + \frac{\partial A}{\partial y} dy \wedge dy \wedge dz + \frac{\partial A}{\partial z} dz \wedge dy \wedge dz + \\
&\quad\ \frac{\partial B}{\partial x} dx \wedge dz \wedge dx + \frac{\partial B}{\partial y} dy \wedge dz \wedge dx + \frac{\partial B}{\partial z} dz \wedge dz \wedge dx + \\
&\quad\ \frac{\partial C}{\partial x} dx \wedge dx \wedge dy + \frac{\partial C}{\partial y} dy \wedge dx \wedge dy + \frac{\partial C}{\partial z} dz \wedge dx \wedge dy \\
&= \quad \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} \right) dx \wedge dy \wedge dz
\end{aligned}$$

where we used $dx \wedge dy = -dy \wedge dx$, $dy \wedge dz = -dz \wedge dy$, etc. and $dx \wedge dx = dy \wedge dy = dz \wedge dz = 0$.

▸ Back

Example Stokes in 3D

Stokes Theorem: Let Ω_{k+1} be a $k+1$-dimensional manifold and $\omega \in \Lambda^k$ then

$$\int_{\partial\Omega_{k+1}} \omega^k = \int_{\Omega_{k+1}} d\omega^k$$

$$k=0: \quad \int_a^b \text{grad}\,\phi\, dx = \phi(b)-\phi(a)\ , \quad \text{grad} \ : \ H_P \longrightarrow H_L$$

$$k=1: \quad \int_{\mathcal{S}} \text{curl}\, A\, dS = \int_{\partial\mathcal{S}} A\, ds\ , \quad \text{curl} \ : \ H_L \longrightarrow H_S$$

$$k=2: \quad \int_{\mathcal{V}} \text{div}\, A\, dV = \int_{\partial\mathcal{V}} A\, dS\ , \quad \text{div} \ : \ H_S \longrightarrow H_V$$

$$\mathbb{R} \hookrightarrow H_P \overset{\text{grad}}{\mapsto} H_L \overset{\text{curl}}{\mapsto} H_S \overset{\text{div}}{\mapsto} H_V \mapsto 0$$

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The action of the coboundary operator I

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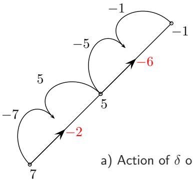
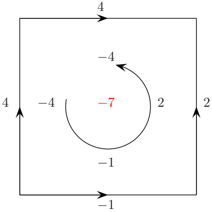
The action of the coboundary operator in pictures



a) Action of δ on 0-chain

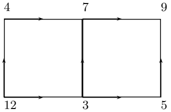


b) Action of δ on 1-chain

The action of the coboundary operator II

▸ Back

Example illustrating the property $\delta\delta c^{(0)} = 0^{(2)}$



given a 0-cochain $c^{(0)} = (12, 3, 5, 9, 7, 4)$

The action of the coboundary operator II

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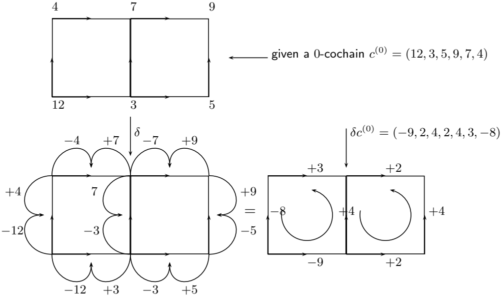
Example illustrating the property $\delta\delta c^{(0)} = 0^{(2)}$



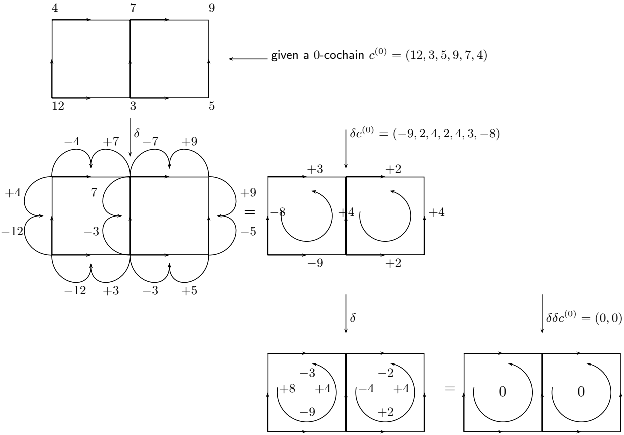
given a 0-cochain $c^{(0)} = (12, 3, 5, 9, 7, 4)$

$\delta c^{(0)} = (-9, 2, 4, 2, 4, 3, -8)$

The action of the coboundary operator II

Back

Example illustrating the property $\delta\delta c^{(0)} = 0^{(2)}$



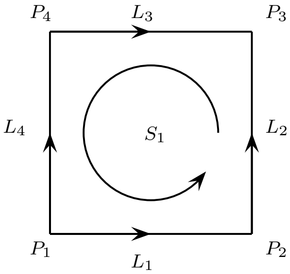
given a 0-cochain $c^{(0)} = (12, 3, 5, 9, 7, 4)$

$\delta c^{(0)} = (-9, 2, 4, 2, 4, 3, -8)$

$\delta\delta c^{(0)} = (0, 0)$

Incidence matrices

The coboundary operator can be represented by a so-called incidence matrix. Consider for instance the cell-complex as shown in the Figure.



K-complex

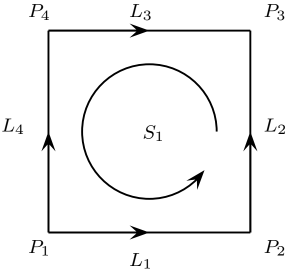
▸ Back

Incidence matrices

The coboundary operator can be represented by a so-called incidence matrix. Consider for instance the cell-complex as shown in the Figure.

$\mathbb{E}^{1,0}$ relates 0-cohains to 1-cochains, i.e "discrete gradient operator"

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$



K-complex

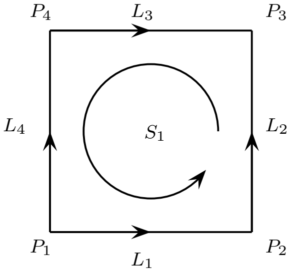
Back

Incidence matrices

The coboundary operator can be represented by a so-called incidence matrix. Consider for instance the cell-complex as shown in the Figure.

$\mathbb{E}^{2,1}$ relates 0-cochains to 1-cochains, i.e "discrete curl operator"

$$\mathbb{E}^{2,1} = \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix}$$



K-complex

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Incidence matrices

The coboundary operator can be represented by a so-called incidence matrix. Consider for instance the cell-complex as shown in the Figure.

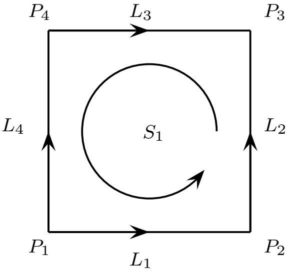
Note that

$$\mathbb{E}^{2,1} \cdot \mathbb{E}^{1,0} =$$

$$\begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

This is matrix expression for $\delta\delta \equiv 0$.



K-complex

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